We first note that this is a regular perturbation problem (why?) and naively try substituting a perturbation series of the form $r(\varepsilon) = r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots$ into (2.3). Note that

$$r^3 = \left(r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots\right)\left(r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots\right)\left(r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots\right)$$
$$= r_0^3 + \varepsilon\left(3r_0^2 r_1\right) + \varepsilon^2\left(3r_0 r_1^2 + 3r_0^2 r_2\right)\ldots$$

so

$$0 = r^3 + r^2 - \varepsilon$$
$$= r_0^3 + \varepsilon\left(3r_0^2 r_1\right) + \varepsilon^2\left(3r_0 r_1^2 + 3r_0^2 r_2\right)\ldots + r_0^2 + \varepsilon\left(2r_0 r_1\right) + \varepsilon^2\left(r_1^2 + 2r_0 r_2\right)\ldots - \varepsilon$$

Collecting powers of ε,

ε^0: $0 = r_0^3 + r_0^2 \quad \Rightarrow \quad r_0^I = -1, \quad r_0^{II,III} = 0$. (unperturbed problem)

ε^1: $0 = 3r_0^2 r_1 + 2r_0 r_1 - 1 \quad \Rightarrow \quad r_1\left(3r_0^2 + 2r_0\right) = 1$.

For $r_0^I = -1$, this implies $r_1^I = \left(3r_0^2 + 2r_0\right)^{-1} = 1$. We can continue the perturbation series for this case.

For $r_0^{II,III} = 0$, $3r_0^2 + 2r_0 = 0$, so the equation for r_1 is inconsistent! This implies that we need to use a different power ε^q, as we'd anticipated.

ε^2: $0 = 3r_0 r_1^2 + 3r_0^2 r_2 + r_1^2 + 2r_0 r_2$

Substituting $r_0 = -1$ and $r_1 = 1$ for root I:

$0 = 3(-1)1^2 + 3(-1)^2 r_2 + (-1)^2 + 2(-1) r_2 \quad \Rightarrow \quad r_2^I = 2$

$$\Rightarrow \quad r^I(\varepsilon) = -1 + \varepsilon + 2\varepsilon^2 \ldots \tag{3.1}$$

For the other two roots, since r_0=0 we try a series of the form $r(\varepsilon) = \varepsilon^q r_1 + \varepsilon^{2q} r_2 \ldots$ Thus r^3=O(ε^{3q}) and r^2=O(ε^{2q}), so

$$0 = r^3 + r^2 - \varepsilon = \mathrm{O}(\varepsilon^{3q}) + \mathrm{O}(\varepsilon^{2q}) - \varepsilon$$

The only way to balance the ε is to make $2q = 1$ (taking $3q = 1$ would lead to an unbalanced O(ε^{2q}) term. Thus, as we suspected from the graph, we must choose $q = 1/2$. Thus

$$0 = r^3 + r^2 - \varepsilon$$
$$= \varepsilon^{3/2} r_1^3 \ldots + \varepsilon r_1^2 + \varepsilon^{3/2}\left(2r_1 r_2\right)\ldots - \varepsilon.$$

Collecting powers of ε,

ε^1: $0 = r_1^2 - 1 \quad \Rightarrow r_1^{II,III} = \pm 1$

$$\varepsilon^{3/2}: \quad 0 = r_1^3 + 2r_1r_2 \quad \Rightarrow \quad r_2^{II,III} = -\frac{r_1^2}{2} = -\frac{1}{2}$$

$$\Rightarrow \quad r^{II,III}(\varepsilon) = \pm\varepsilon^{1/2} - \varepsilon/2\ldots \qquad (3.2)$$

Fig. 3.1 plots the first two partial sums of the perturbation series for the three roots vs. ε for the range $0 < \varepsilon < 0.15$, in which they are all real. They are compared with exact roots computed using the Matlab `roots` function. In this case, the perturbation series start losing accuracy for $\varepsilon \approx 0.1$.

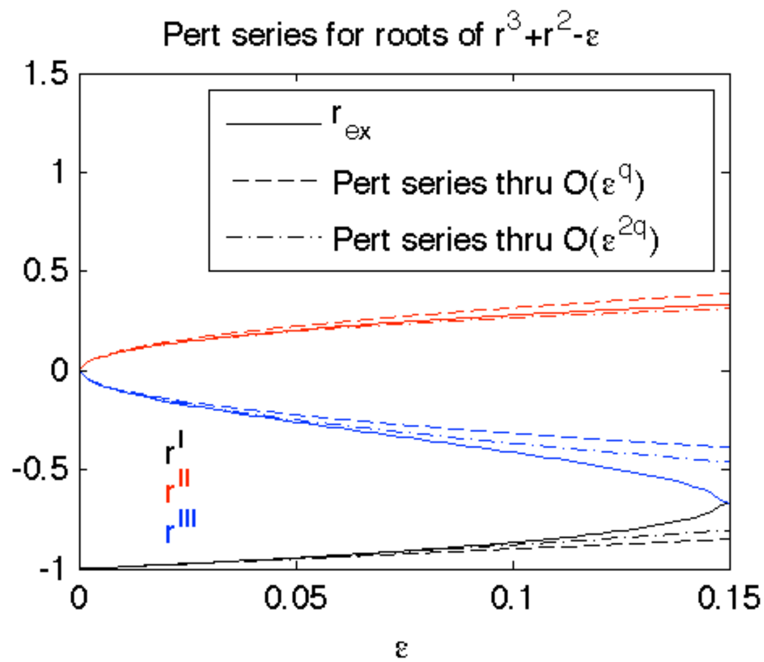

Fig. 3.1: Convergence of perturbation series (3.1) and (3.2) for the three roots of the cubic.