Example (R2): Find a perturbation series solution of $y'' + \varepsilon y' = 1, \quad y(0) = 1,\ y(1) = 0,\ \varepsilon \ll 1.$

Note: This problem has a closed-form exact solution which is a particular solution (which we can guess in this case) plus a linear combination of homogeneous solutions:

$$y_E(x) = x/\varepsilon + c_1 e^{-\varepsilon x} + c_2$$

The coefficients are determined from the boundary conditions:

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ 0 &= y(1) = \varepsilon^{-1} + c_1 e^{-\varepsilon} + c_2 \end{aligned} \quad \Rightarrow \quad \begin{cases} c_1 = (1+\varepsilon^{-1})/(1-e^{-\varepsilon}) \\ c_2 = 1 - c_1 \end{cases}$$

This solution can be used to test the convergence of our perturbation solution.

Solution: Guess a series of the form $y(x,\varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x)\ldots$ and substitute into (R2):

$$y_0''(x) + \varepsilon y_1''(x) + \varepsilon^2 y_2''(x)\ldots + \varepsilon\left\{ y_0'(x) + \varepsilon y_1'(x) + \varepsilon^2 y_2'(x)\ldots \right\} = 1$$

$$y(0) = y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0)\ldots = 1 = 1 + 0\varepsilon + 0\varepsilon^2\ldots$$

$$y(1) = y_0(1) + \varepsilon y_1(1) + \varepsilon^2 y_2(1)\ldots = 0 = 0 + 0\varepsilon + 0\varepsilon^2\ldots$$

Sorting into powers of ε:

ε^0: $y_0''(x) = 1, \quad y_0(0) = 1,\ y_0(1) = 0. \quad \Rightarrow y_0(x) = \dfrac{x^2}{2} - \dfrac{3x}{2} + 1$

ε^1: $y_1''(x) + y_0'(x) = 0 \quad \Rightarrow y_1''(x) = -y_0'(x) = -x + \dfrac{3}{2} \quad \Rightarrow y_1(x) = -\dfrac{x^3}{6} + \dfrac{3x^2}{4} + c_3 x + c_4$

$$y_1(0) = 0 \quad \Rightarrow \quad 0 = -\frac{1}{6}0^3 + \frac{3}{4}0^2 + c_3 0 + c_4 \quad \Rightarrow \quad c_4 = 0$$

$$y_1(1) = 0 \quad \Rightarrow \quad 0 = -\frac{1}{6}1^3 + \frac{3}{4}1^2 + c_3 1 + c_4 \quad \Rightarrow c_3 = -\frac{7}{12} - c_4 = -\frac{7}{12}$$

$$\Rightarrow y_1(x) = -\frac{x^3}{6} + \frac{3x^2}{4} - \frac{7x}{12}$$

ε^2: $y_2''(x) + y_1'(x) = 0 \quad \Rightarrow y_2''(x) = -y_1'(x) = \dfrac{x^2}{2} - \dfrac{3}{2}x + \dfrac{7}{12} \quad \Rightarrow y_2(x) = \dfrac{x^4}{24} - \dfrac{x^3}{4} + \dfrac{7x^2}{24} + c_5 x + c_6$

$$y_2(0) = 0 \quad \Rightarrow \quad 0 = \frac{1}{24}0^4 - \frac{1}{4}0^3 + \frac{7}{24}0^2 + c_5 0 + c_6 \quad \Rightarrow \quad c_6 = 0$$

$$y_2(1) = 0 \quad \Rightarrow \quad 0 = \frac{1}{24}1^4 - \frac{1}{4}1^3 + \frac{7}{24}1^2 + c_5 1 + c_6 \quad \Rightarrow \quad c_5 = -\frac{1}{12}$$

$$\Rightarrow y_2(x) = \frac{x^4}{24} - \frac{x^3}{4} + \frac{7x^2}{24} - \frac{x}{12}$$

The upper left panel of Fig. 4.1 shows the exact solutions for ε = 0, 1, 2, 3. Although ε is apparently quite large, we can see that the exact solutions are varying smoothly and fairly linearly with ε, suggesting a perturbation series in powers of ε might be quite accurate.

The remaining panels plot the error of the 0^{th}, 1^{st}, and 2^{nd} –order perturbation series approximations at each *x*. The 0^{th}-order series (upper right) has max error proportional to ε, as we expect since this is the order of the first neglected term in the perturbation series. Analogously, the 1^{st} -order series (lower left) has max error proportional to ε^2, and the 2^{nd} –order series has max error proportional to ε^2. For ε = 1 (the green curve), the max error of the 2^{nd} –order perturbation series is 0.0024, which is remarkably small. This shows that indeed, a perturbation series gives an excellent approximation to the exact solution of this BVP over a broad range of ε.

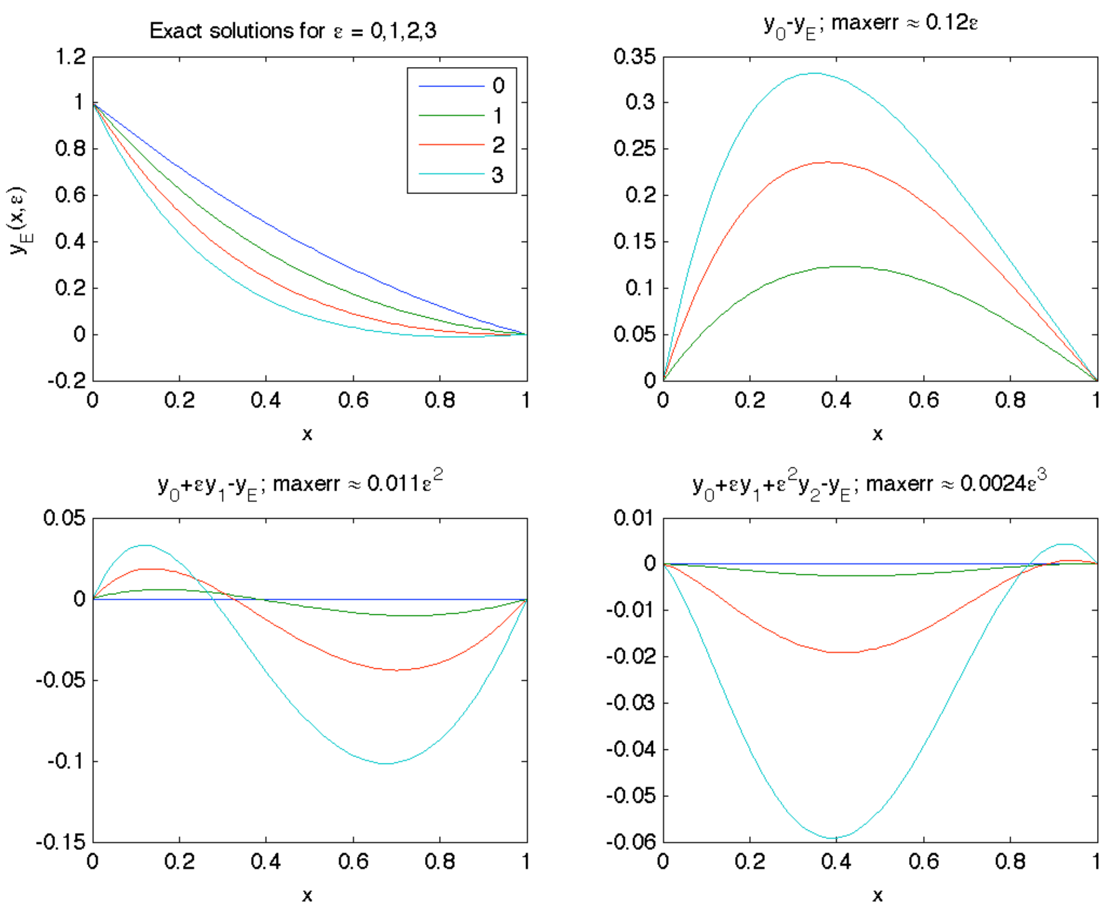

Fig. 4.1: Exact solution and error of perturbation series through orders 0-2 for BVP (R2), ε=0,1,2,3.