Approach 2: Integral for nonlinear pendulum period based on potential theory

This approach exploits the mathematical structure of the nonlinear problem to reduce the amount of computation needed to deduce the period. Multiply the nonlinear pendulum equation $\theta'' + \sin\theta = 0$ by θ', and integrate with respect to t to get

$$H(\theta,\theta') = \theta'^2 / 2 - \cos\theta = E = -\cos\varepsilon \tag{6.1}$$

The functional $H(\theta,\theta')$, the Hamiltonian, can be interpreted as a sum of nondimensionalized kinetic and potential energy. The constant of integration E is the total energy, which we have determined from the initial conditions. We solve (6.1) for θ':

$$d\theta / dt = \theta' = \pm\left\{2\left(\cos\theta - \cos\varepsilon\right)\right\}^{1/2} \tag{6.2}$$

We separate variables in (6.2). We integrate over the quarter-period of the pendulum downswing from θ = ε to θ = 0, during which $\theta' < 0$, so we use the '-' branch of the square root:

$$\int_0^{T/4} dt = -\int_\varepsilon^0 \left\{2\left(\cos\theta - \cos\varepsilon\right)\right\}^{-1/2} d\theta$$

$$T(\varepsilon) = 4\int_0^\varepsilon \left\{2\left(\cos\theta - \cos\varepsilon\right)\right\}^{-1/2} d\theta \tag{6.3}$$

In general, the expression for T is a *complete elliptic integral* which can only be evaluated numerically. However, we can approximately find T for small ε, as follows. We know that in the integral, θ is O(ε). Thus in (6.3) we can use Taylor series approximations of cos θ and cos ε:

$$\cos y = 1 - \frac{y^2}{2} + \frac{y^4}{24}\ldots, \quad \cos\varepsilon = 1 - \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{24}\ldots$$

$$\Rightarrow T(\varepsilon) = 4\int_0^\varepsilon \left\{\left(\varepsilon^2 - \theta^2\right) - \frac{1}{12}\left(\varepsilon^4 - \theta^4\right)\ldots\right\}^{-1/2} d\theta = 4\int_0^\varepsilon \left(\varepsilon^2 - \theta^2\right)^{-1/2} \left\{1 - \frac{1}{12}\left(\varepsilon^2 + \theta^2\right)\ldots\right\}^{-1/2} d\theta$$

In this integral, set $\theta = \varepsilon\sin u$, so sin u runs from 0 to 1 and u runs from 0 to π/2:

$$T = 4\int_0^{\pi/2} \left(\varepsilon^2\cos^2 u\right)^{-1/2} \left\{1 - \frac{\varepsilon^2}{12}\left(1 + \sin^2 u\right)\ldots\right\}^{-1/2} \varepsilon\cos u \, du$$

$$= 4\int_0^{\pi/2} \left\{1 - \frac{\varepsilon^2}{12}\left(1 + \sin^2 u\right)\ldots\right\}^{-1/2} du = 4\int_0^{\pi/2} \left\{1 + \frac{\varepsilon^2}{24}\left(1 + \sin^2 u\right)\ldots\right\} du$$

$$= 4\frac{\pi}{2}\left\{1 + \frac{\varepsilon^2}{24}\left(1 + \frac{1}{2}\right)\ldots\right\} = 2\pi\left\{1 + \frac{\varepsilon^2}{16}\ldots\right\},$$

just as we deduced from the brute force method.