Method of Multiple Scales applied to nonlinear pendulum problem

The perturbation series that we derived for $\theta(t)$ had a secular term (proportional to t) that causes it to lose validity at large times, even for small ε. The physical cause of this problem was a slow phase drift of the true solution away from the unperturbed solution due to the nonlinear correction to the pendulum period. It would be nice to devise a perturbation series approach that remained valid at long time by accounting for such slow drifts to as to avoid secular terms. The *method of multiple scales* accomplishes this goal.

The idea, as applied to the nonlinear pendulum problem,

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0, \quad \theta(0) = \varepsilon \ll 1, \frac{d\theta}{dt}(0) = 0. \tag{P}$$

is to make $\theta(t)$ formally a function of multiple time scales that capture the fast and slow time variability of the solution. Since the first nonlinear correction to the solution is of order ε^2 compared to the unperturbed solution, we introduce the timescales

$$t_0 = t, \;\; t_2 = \varepsilon^2 t, \; ...$$

and (to the order of accuracy needed for our purposes)

$$\theta(t) = \varepsilon\theta_1(t_0\,, t_2) + \varepsilon^3\theta_3(t_0\,, t_2) + \ldots$$

Having multiple variables to describe the time dependence is redundant, but this redundancy can be exploited to develop a perturbation series with no secular terms that is uniformly valid for all small ε. Turning to (P), note that

$$\frac{d\theta_k}{dt} = \frac{\partial\theta_k}{\partial t_0}\frac{dt_0}{dt} + \frac{\partial\theta_k}{\partial t_2}\frac{dt_2}{dt} = \frac{\partial\theta_k}{\partial t_0} + \varepsilon^2\frac{\partial\theta_k}{\partial t_2} \tag{6a.1}$$

so that

$$\frac{d^2\theta_k}{dt^2} = \left(\frac{\partial}{\partial t_0} + \varepsilon^2\frac{\partial}{\partial t_2}\right)^2 \theta_k = \left(\frac{\partial^2}{\partial t_0^2} + 2\varepsilon^2\frac{\partial^2}{\partial t_0 \partial t_2} + ...\right)\theta_k \tag{6a.2}$$

while

$$\sin\theta = \varepsilon\theta_1 + \varepsilon^3(\theta_3 - \theta_1^3/6)\ldots$$

Substituting these series into (P) and sorting by increasing powers of ε, we get:

$$\varepsilon^1\!: \;\; 0 = \frac{\partial^2\theta_1}{\partial t_0^2} + \theta_1, \quad \theta_1(t_0 = 0, t_2 = 0) = 1, \quad \frac{\partial\theta_1}{\partial t_0}(t_0 = 0, t_2 = 0) = 0. \tag{6a.3}$$

$$\Rightarrow \quad \theta_1(t) = A(t_2)\cos(t_0) + B(t_2)\sin(t_0), \quad A(0) = 1, B(0) = 0 \tag{6a.4}$$

The formal dependence on the slow time t_2 allows the leading-order solution to be more general than in our original perturbation series solution. This is how the method will account for the slow phase shift of the perturbed solution. Going on to the next power of ε:

$$\varepsilon^3\text{:}\ \frac{\partial^2\theta_3}{\partial t_0^2} + 2\frac{\partial^2\theta_1}{\partial t_0 \partial t_2} + \theta_3 - \frac{\theta_1^3}{6} = 0, \quad \text{ICs: } \theta_3 = \frac{\partial\theta_3}{\partial t_0} + \frac{\partial\theta_1}{\partial t_2} = 0 \text{ for } t_0 = t_2 = 0 \tag{6a.5}$$

Substituting the known form for θ_1, we deduce

$$\frac{\partial^2\theta_3}{\partial t_0^2} + \theta_3 = -2\frac{\partial^2\theta_1}{\partial t_0 \partial t_2} + \frac{\theta_1^3}{6} = 2\frac{dA}{dt_2}\sin t - 2\frac{dB}{dt_2}\cos t + \frac{1}{6}\left(A\cos t + B\sin t\right)^3 \tag{6a.6}$$

$$\text{ICs: } \theta_3(0,0) = \frac{\partial\theta_3}{\partial t_0}(0,0) + \frac{dA_2}{dt_2}(0) = 0 \tag{6a.7}$$

We write the RHS of (6a.6) as a linear combination of terms involving e^{it}, e^{-it}, e^{3it}, e^{-3it} by noting

$$2\frac{dA}{dt_2}\sin t - 2\frac{dB}{dt_2}\cos t = 2\frac{dA}{dt_2}\left(\frac{e^{it} - e^{-it}}{2i}\right) - 2\frac{dB}{dt_2}\left(\frac{e^{it} + e^{-it}}{2}\right)$$

$$= -e^{it}\left(\frac{dB}{dt_2} + i\frac{dA}{dt_2}\right) - e^{-it}\left(\frac{dB}{dt_2} - i\frac{dA}{dt_2}\right)$$

$$\left(A\cos t + B\sin t\right)^3 = \frac{1}{8}\left([A - iB]e^{it} + [A + iB]e^{-it}\right)^3$$

$$= \frac{1}{8}\left([A - iB]^3 e^{3it} + 3[A - iB]^2[A + iB]e^{it} + 3[A - iB][A + iB]^2 e^{-it} + [A + iB]^3 e^{-3it}\right) \tag{6a.8}$$

To suppress secular terms in $\theta_3(t_0)$, the coefficient of e^{it} and e^{-it} in (6a.6) must be zero:

$$e^{it}\text{:}\ 0 = -\left(\frac{dB}{dt_2} + i\frac{dA}{dt_2}\right) + \frac{3}{48}[A - iB]^2[A + iB] \tag{6a.9}$$

The e^{-it} term is just the complex conjugate of this, so it yields no additional information. Noting $\{A-iB][A+iB] = A^2 + B^2$ and separating into real and imaginary parts, this yields

$$\frac{dB}{dt_2} = \frac{1}{16}A(A^2 + B^2); \quad \frac{dA}{dt_2} = -\frac{1}{16}B(A^2 + B^2) \tag{6a.10}$$

This pair of coupled ODEs determines the slow evolution of the leading-order solution. It looks scary, but by adding B times the first ODE to A times the second, we conclude that

$$0 = B\frac{dB}{dt_2} + A\frac{dA}{dt_2} = \frac{1}{2}\frac{d}{dt_2}\left(A^2 + B^2\right) \quad \Rightarrow \quad A^2 + B^2 = \text{constant} = A^2(0) + B^2(0) = 1 \tag{6a.11}$$

This allows the ODEs (6a.10) to simplify to those for circular motion:

$$\frac{dB}{dt_2} = \frac{1}{16}A; \quad \frac{dA}{dt_2} = -\frac{1}{16}B; \quad \text{with IC } A(0) = 1, B(0) = 0. \tag{6a.12}$$

The solution to (6a.12) is

$$A(t_2) = \cos\left(\frac{t_2}{16}\right); \quad B(t_2) = \sin\left(\frac{t_2}{16}\right) \tag{6a.13}$$

We can now deduce the slow-time behavior of the leading-order solution:

$$\theta_1(t) = A(t_2)\cos(t_0) + B(t_2)\sin(t_0) = \cos\left(\frac{t_2}{16}\right)\cos(t_0) + \sin\left(\frac{t_2}{16}\right)\sin(t_0)$$

$$= \cos\left(t_0 - \frac{t_2}{16}\right) = \cos t\left(1 - \frac{\varepsilon^2}{16}\right) \tag{6a.14}$$

The corresponding period is

$$T = \frac{2\pi}{1-\varepsilon^2/16} \approx 2\pi\left(1+\varepsilon^2/16\right), \tag{6a.15}$$

which is reassuringly in agreement with the two methods described in the previous two lectures.

If desired, we can now deduce the fast-time behavior of the first correction term $\theta_3(t_0, t_2)$ from (6a.6-8), noting that the remaining terms on the RHS are linear combinations of e^{3it}, e^{-3it}:

$$\frac{\partial^2\theta_3}{\partial t_0^2} + \theta_3 = \frac{1}{48}\left([A-iB]^3 e^{3it_0} + [A+iB]^3 e^{-3it_0}\right) = \frac{1}{48}\left(e^{-3it_2/16}e^{3it_0} + e^{3it_2/16}e^{-3it_0}\right) \tag{6a.16}$$

with $\theta_3(0,0) = 0; \quad \frac{\partial\theta_3}{\partial t_0}(0,0) = -\frac{dA}{dt_2}(0) = 0$

By guessing a particular solution that is a linear combination of e^{3it}, e^{-3it}, we deduce that

$$\theta_3(t_0,t_2) = -\frac{1}{192}\cos 3\left(t_0 - t_2/16\right) + A_3(t_2)\cos t_0 + B_3(t_2)\sin t_0 \tag{6a.17}$$

Using the homogeneous solutions to match the ICs, we deduce that $A_3(t_2) = 1/192$, $B_3(0) = 0$. We would have to go to next order to show that

$$\theta_3(t_0,t_2) = \frac{1}{192}\left(\cos[t_0 - t_2/16] - \cos 3[t_0 - t_2/16]\right), \tag{6a.18}$$

consistent with (5.5), which we derived from the regular perturbation series with some hand-waving.