Parametric Resonance and Pumping a Swing

For our last example of regular perturbation analysis, we consider a problem related to how a child 'pumps' a swing to make it go higher by alternatively swinging his legs forward as he swings forward, then pulls them back as he swings back.

We idealize this problem as a mass *m* swinging on the end of a frictionless pendulum whose length $r(\tau) = r_0\{1 + \varepsilon \sin(\omega\tau + \alpha)\}$ is sinusoidally varied as a function of time τ with frequency ω and initial phase α. We assume $\varepsilon \ll 1$. We let $\theta(\tau)$ be the counterclockwise angle of the pendulum from the vertical. We obtain an ODE for the pendulum motion by equating the rate of change of angular momentum $mr^2\theta'(\tau)$ on the pendulum to the gravitational torque $-mgr \sin\theta$ on it:

$$(mr^2\theta')' = -mgr \sin\theta$$

We assume that $\theta \ll 1$, so $\sin\theta \approx \theta$, yielding the ODE

$$r\theta'' + 2r'\theta' + g\theta = 0.$$

By defining $\theta(\tau) = r\theta/r_0$ and substituting into the ODE, we obtain

$$\phi'' = r\theta'' + 2r'\theta' + r''\theta = -g\theta + r''\theta = -\frac{g - r''}{r}\phi$$

This is just like the conventional linear pendulum equation except *r* is time-varying and gravity *g* is reduced by the downward acceleration from the lengthening of the pendulum.

We can simplify the right hand side using the assumption $\varepsilon \ll 1$:

$$\frac{g - r''}{r} = \frac{g + \varepsilon r_0\omega^2 \sin(\omega\tau + \alpha)}{r_0\{1 + \varepsilon \sin(\omega\tau + \alpha)\}} = \omega_0^2\left\{\frac{1 + \varepsilon\sigma^2 \sin(\omega\tau + \alpha)}{1 + \varepsilon \sin(\omega\tau + \alpha)}\right\} \approx \omega_0^2\left\{1 + \varepsilon\left(\sigma^2 - 1\right)\sin(\omega\tau + \alpha)\right\}$$

where $\omega_0 = (g/r_0)^{1/2}$ is the natural pendulum frequency and $\sigma = \omega/\omega_0$ is the ratio of the length oscillation frequency to the natural pendulum frequency.

Lastly, we nondimensionalize the ODE by introducing a nondimensional time $t = \omega_0\tau$ and defining $y(t) = \theta(\tau)$:

$$y'' + \left\{1 + \varepsilon\left(\sigma^2 - 1\right)\sin(\sigma t + \alpha)\right\}y = 0, \quad \varepsilon \ll 1, \quad \sigma = O(1). \tag{M}$$

This is called a *Mathieu equation*; the same ODE arises in circuit analysis and other applications. Let's use regular perturbation theory to see what happens if we start the pendulum from rest at some amplitude (which we can take to be 1 since this is a linear problem), i. e.

$$y(0)=1, \quad y'(0)=0.$$

We try a solution of the form $y(t) = y_0(t) + \varepsilon y_1(t)\ldots$ Substituting into (M) and the initial conditions and sorting powers of ε:

$$\varepsilon^0:\ y_0'' + y_0 = 0, \quad y_0(0) = 1, y_0'(0) = 0 \quad \Rightarrow \quad y_0(t) = \cos t$$

$$\varepsilon^1:\ y_1'' + y_1 = -\left(\sigma^2 - 1\right)\sin(\sigma t + \alpha) y_0, \quad y_1(0) = 0, y_1'(0) = 0.$$

The O(ε) problem can be solved with a Laplace transform. We use the trigonometric identity

$$2\sin(\sigma t + \alpha)\cos t = \sin(\left[\sigma + 1\right]t + \alpha) + \sin(\left[\sigma - 1\right]t + \alpha).$$

The algebra is simplified by solving

$$Y'' + Y = \frac{1-\sigma^2}{2}\left\{\exp i(\left[\sigma+1\right]t + \alpha) + \exp i(\left[\sigma - 1\right]t + \alpha)\right\}, \quad Y(0) = Y'(0) = 0 \tag{M1}$$

and taking $y_1(t) = \mathrm{Im}\, Y(t)$ at the end. We define the Laplace transform of Y,

$\tilde{Y}(s) = L[Y(t)] = \int_0^\infty Y(t)e^{-st}dt$, and take the Laplace transform of (M1):

$$(s^2+1)\tilde{Y}(s) = \frac{1-\sigma^2}{2}\int_0^\infty \left(e^{i[\sigma+1]t + i\alpha} + e^{i[\sigma-1]t+i\alpha}\right)e^{-st}dt$$

$$\Rightarrow \quad \tilde{Y}(s) = \frac{\left(1-\sigma^2\right)e^{i\alpha}}{2(s+i)(s-i)}\left\{\frac{1}{s - i\left[\sigma+1\right]} + \frac{1}{s - i\left[\sigma - 1\right]}\right\} = \frac{\left(1-\sigma^2\right)e^{i\alpha}}{(s+i)(s-i)}\left\{\frac{s - i\sigma}{\left(s - i\left[\sigma+1\right]\right)\left(s - i\left[\sigma-1\right]\right)}\right\}$$

We can use the Mellin inversion formula for the Laplace transform:

$$Y(t) = \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} \underbrace{\tilde{Y}(s)e^{st}}_{\Psi(s)} ds$$

where $A > 0$ is chosen so that this contour stays to the right of the singularities of $\tilde{y}(s)$, which are poles at $s_k = \pm i$ and $i(\sigma \pm 1)$. The contour is closed without additional contribution in the left half-plane and evaluated using residue theory as $Y(t) = \sum_k \mathrm{Res}_\Psi(s_k)$.

If $\sigma \neq 0, 2$, all the poles are simple and $\mathrm{Res}_\Psi(s_k) = \lim_{s \to s_k}(s - s_k)\tilde{Y}(s)e^{st}$. In this case

$$Y(t) = a_1 e^{it} + a_2 e^{-it} + a_3 e^{i(\sigma+1)t} + a_4 e^{i(\sigma-1)t}, \quad a_k = \mathrm{Res}_{\tilde{Y}} s_k$$

In this case, the solution $y_1(t) = \mathrm{Im}\, Y(t)$ is a sum of sinusoids, which will remain bounded. Thus, the oscillation of the pendulum length does not systematically pump up the amplitude of the pendulum swing. The case $\sigma = 0$ corresponds to a pendulum whose length remains constant at $r_0\{1 + \varepsilon \sin \alpha\}$. In this case, the pendulum swing amplitude will remain the same but its period will be slightly perturbed, causing a systematic drift of the pendulum phase compared to the unperturbed problem that manifests itself as a secular term in the O(ε) problem.

The most interesting case, called *parametric resonance*, is σ = 2, in which the pendulum length is oscillated at twice the natural pendulum frequency. For this case there is a double pole at $s_1 = i$ and simple poles at $s_2 = -i$ and $s_3 = 3i$. The residue at the double pole is

$$\mathrm{Res}_\Psi(i) = \frac{d}{ds}\left\{(s-i)^2\Psi\right\}\Big|_{s=i} = -3e^{i\alpha}\frac{d}{ds}\left\{\frac{s-2i}{(s+i)(s-3i)}e^{st}\right\}\Bigg|_{s=i}$$

$$= -3e^{i\alpha}e^{it}\left[\frac{d}{ds}\left\{\frac{s-2i}{(s+i)(s-3i)}\right\} + t\left\{\frac{s-2i}{(s+i)(s-3i)}\right\}\right]_{s=i}$$

Now,

$$\frac{s-2i}{(s+i)(s-3i)} = \frac{3/4}{s+i} + \frac{1/4}{s-3i} \quad \Rightarrow \quad \frac{d}{ds}\frac{s-2i}{(s+i)(s-3i)} = -\frac{3}{4}\frac{1}{(s+i)^2} - \frac{1}{4}\frac{1}{(s-3i)^2},$$

so

$$\mathrm{Res}_\Psi(i) = -3e^{i(t+\alpha)}\left[\left\{-\frac{3}{4}\frac{1}{(i+i)^2} - \frac{1}{4}\frac{1}{(i-3i)^2}\right\} + t\frac{i-2i}{(i+i)(i-3i)}\right] = -3e^{i(t+\alpha)}\left[1-it\right]/4$$

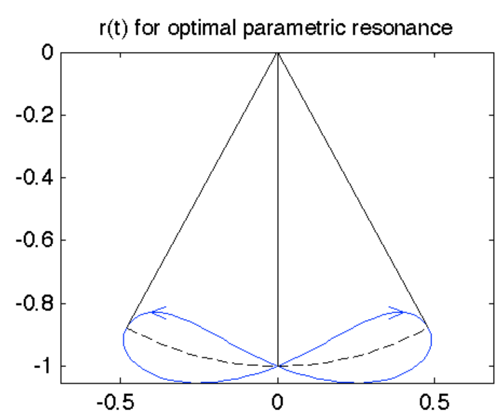
Thus the solution has the form

$$Y(t) = -3e^{i(t+\alpha)}\left[1-it\right]/4 + a_2e^{-it} + a_3e^{3it}, \quad a_{2,3} = \mathrm{Res}_{\tilde{Y}}s_{2,3} \quad (\sigma = 2)$$

There is a secular term $Y_{\mathrm{sec}}(t) = 3ite^{i(t+\alpha)}/4$ plus oscillatory terms. Taking the imaginary part to recover the secular term in the physical solution,

$$y_{1,\mathrm{sec}}(t) = \mathrm{Im}\,Y_{\mathrm{sec}}(t) = 3t\cos(t+\alpha)/4$$

so $$y(t) = y_0(t) + \varepsilon y_1(t)\ldots = \cos t + 3\varepsilon t\cos(t+\alpha)/4$$
$$+O(\varepsilon) \text{ oscillatory terms}$$



The swing amplitude increases fastest if α = 0, i. e. if

$$r(\tau) = r_0\{1 + \varepsilon\sin(2\omega_0\tau)\}.$$

Thus we should lengthen the pendulum at it goes through the top of each upswing and shorten it as it goes through the bottom of each swing. This is similar, but not identical, to what a child does on a swing, because of the angular momentum of her legs swinging around her body also matters there (see http://retro.grinnell.edu/academic/physics/faculty/case/swing). Nevertheless the basic idea of inducing parametric resonance still applies to the real case of pumping a swing.