Singular Perturbation Theory

Recall that a singular perturbation involves a small parameter ε which changes the order or degree of the problem so as to introduce a new class of solutions not present in the unperturbed problem.

The general strategy is:

(1) Use the *Method of Dominant Balance* (described below, using an example) to find a 'leading-order' approximation $y_0(x;\varepsilon)$ which *asymptotically* approaches the true solution $y(x)$ as $\varepsilon \to 0$, denoted

$$y_0(x;\varepsilon) \sim y(x) \text{ as } \varepsilon \to 0\,,$$

which means

$$\lim_{\varepsilon\to 0} \frac{y(x;\varepsilon)}{y_0(x;\varepsilon)} = 1 \quad \text{for all } x$$

For instance $\sin \varepsilon x \sim \varepsilon x$ as $\varepsilon \to 0$.

For a regular perturbation problem this would just be the solution to the unperturbed problem, but there will be additional solutions in a singular perturbation problem. If the problem has no x dependence, just ignore the x's in the above.

(2) If more accuracy is desired, seek a perturbation series solution, which usually takes the form:

$$y(x;\varepsilon) \sim y_0(x;\varepsilon)\left\{1+\varepsilon^q y_1(x)+\varepsilon^{2q} y_2(x)\ldots\right\}$$

This is an *asymptotic series*. That is , if we let $y_{\Sigma n} = y_0(x;\varepsilon)\left\{1+\varepsilon^q y_1(x)+\varepsilon^{2q} y_2(x)\ldots+\varepsilon^{nq} y_n(x)\right\}$ be the n'th partial sum of the series,

$$\lim_{\varepsilon\to 0} \frac{y(x;\varepsilon)-y_{\Sigma n}(x;\varepsilon)}{y_n(x;\varepsilon)} = 0$$

An asymptotic series need not converge for any $\varepsilon \to 0$! We'll see some good examples for this later when we find asymptotic series for the solutions of ODEs around irregular singular points, but for now I'll leave you to wonder how this could be, and how such series could be useful.

Example S1

Find perturbation series for the roots of

$$p(r;\varepsilon) = \underbrace{\varepsilon r^3}_{A} + \underbrace{r^2}_{B} - \underbrace{1}_{C}, \quad \varepsilon \ll 1 \tag{S1}$$

This is a singular perturbation problem because the unperturbed polynomial is quadratic and has only two roots $r_0^{(1,2)} = \pm 1$ (for which we can develop regular perturbation series), while the perturbed problem has three roots.

Exercise: Show that the perturbation series for these two roots are $r^{(1,2)} = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{5}{8}\varepsilon^2 \ldots$

The asymptotic form of the third root $r^{(3)}$ is derived using dominant balance. We label the three terms of the polynomial (S1) as A, B and C. As $\varepsilon \to 0$, at least two of these terms have to be of the same order in ε so they can balance each other. This is a *consistent* dominant balance if all the other terms are be much smaller as $\varepsilon \to 0$. For instance, If we assume a dominant balance between B and C, we deduce that $r^2 \sim 1$, which yields the unperturbed roots. This balance is consistent, since if $r \sim \pm 1$, then $A = O(\varepsilon) << 1$.

If we assume a dominant balance between A and C, we deduce $r \sim O(\varepsilon^{-1/3})$, for which both A and C are O(1). This would imply term $B = r^2 = O(\varepsilon^{-2/3}) >> A, C$, so this dominant balance is inconsistent.

If we assume a dominant balance between A and B, we deduce $r \sim -\varepsilon^{-1}$. In this case, A and B are $O(\varepsilon^{-2})$ while C is $O(1) << A, B$, so this balance is consistent. Hence

$$r^{(3)} \sim r_0^{(3)} = -\frac{1}{\varepsilon}$$

In most circumstances, the dominant balance in a singular perturbation problem comes from balancing the two highest-order terms, as we found here. An exception could occur if the coefficient of the second highest order term also depends on ε and goes to zero as $\varepsilon \to 0$.

This *leading-order approximation* may already be all we need to know about the third root. But if desired we can find an asymptotic series $r^{(3)} \sim -\frac{1}{\varepsilon}\left\{1 + r_1\varepsilon^q + r_2\varepsilon^{2q} \ldots\right\}$. Substituting this form into (S1) and sorting powers of ε, we obtain:

$$\varepsilon\left(r^{(3)}\right)^3 \sim \varepsilon\left(-\frac{1}{\varepsilon}\right)^3 \left\{1 + \varepsilon^q(3r_1) + \varepsilon^{2q}(3r_2 + 3r_1^2)\ldots\right\}$$

$$\left(r^{(3)}\right)^2 \sim \left(-\frac{1}{\varepsilon}\right)^2 \left\{1 + \varepsilon^q(2r_1) + \varepsilon^{2q}(2r_2 + r_1^2)\ldots\right\}$$

$$\Rightarrow \quad 0 = \varepsilon\left(r^{(3)}\right)^3 + \left(r^{(3)}\right)^2 - 1$$

$$=\frac{1}{\varepsilon^2}\left\{-1-\varepsilon^q(3r_1)-\varepsilon^{2q}(3r_2+3r_1^2)\ldots+1+\varepsilon^q(2r_1)+\varepsilon^{2q}(2r_2+r_1^2)\ldots\right\}-1$$

$$=-\varepsilon^{q-2}r_1-\varepsilon^{2q-2}(r_2+2r_1^2)\ldots-1$$

To match powers of ε, we must have $q = 2$ and $r_1 = -1$.

The next power of ε to balance is:

$O(\varepsilon^2)$: $\quad 0 = -(r_2 + 2r_1^2) \;\Rightarrow\; r_2 = -2r_1^2 = -2.$

Hence

$$r^{(3)} \sim -\frac{1}{\varepsilon}\left\{1-\varepsilon^2-2\varepsilon^4\ldots\right\}$$

<u>Accuracy</u> (using Matlab roots function to find exact $r^{(3)}$):

ε	$r_{\Sigma 2}^{(3)}$	$r_{ex}^{(3)} - r_{\Sigma 2}^{(3)}$	$\frac{r_{ex}^{(3)} - r_{\Sigma 2}^{(3)}}{\varepsilon^5}$
0.05	-19.95	$-2.2\text{x}10^{-6}$	-7.1
0.1	-9.90	$-7.3\text{x}10^{-5}$	-7.3
0.2	-4.78	$-2.7\text{x}10^{-3}$	-8.5
0.3	-2.98	-0.03	-11.9

We see that the partial sum given above gives a relative accuracy of better than 1% even for ε = 0.3. It is accurate to $O(\varepsilon^5)$ (the next term in the expansion), which goes to zero faster than the last term retained in the series, which is $O(\varepsilon^3)$. This is consistent with the definition of an asymptotic series.