

### Singularly Perturbed Boundary-Value Problems and Boundary Layers

Example S2: Find an approximate solution to

$$\underbrace{\varepsilon y''}_A + \underbrace{y'}_B = \underbrace{1}_C, \quad \underbrace{y(0)=1}_{BC1}, \underbrace{y(1)=0}_{BC1}, \quad 0 < \varepsilon \ll 1. \quad (S2)$$

This is a singular perturbation problem since the unperturbed problem is a 1<sup>st</sup>-order ODE and the perturbed problem is a 2<sup>nd</sup>-order ODE with an extra linearly independent solution.

As always, we start by considering the unperturbed ODE

$$y'_0 = 1 \Rightarrow y_0(x) = x + c_0$$

Regardless of  $c_0$ , this solution cannot satisfy both BCs. However, it will still be very useful!

This BVP is a nice learning case because it has a simple exact solution of the form

$$y_{ex}(x; \varepsilon) = x + c_0 + c_1 e^{-x/\varepsilon}$$

Define  $\delta = \exp(-1/\varepsilon)$ . To satisfy the BCs,

$$(BC1) \Rightarrow 1 = y(0) = c_0 + c_1$$

$$(BC2) \Rightarrow 0 = y(1) = 1 + c_0 + c_1 \delta \Rightarrow c_1 = \frac{2}{1-\delta}, c_0 = -\frac{1+\delta}{1-\delta}$$

For  $0 < \varepsilon \ll 1$ ,  $\delta$  is extremely small, so  $c_1 \approx 2$ ,  $c_0 \approx -1$ , and

$$y(x; \varepsilon) \approx x - 1 + 2e^{-x/\varepsilon} \approx \begin{cases} -1 + 2e^{-x/\varepsilon}, & 0 \leq x \ll 1 \quad \text{'inner' solution} \\ x - 1, & x \gg \varepsilon \quad \text{'outer' solution} \end{cases} \quad (9.1)$$

Adjacent to the boundary  $x = 0$ ,  $y(x)$  changes rapidly across a *boundary layer* of thickness  $O(\varepsilon)$ .

As an interesting aside, were  $\varepsilon < 0$ , but still small, then  $\delta \gg 1$ ,  $c_1 \approx -2e^{-1/\varepsilon}$ ,  $c_0 \approx 1$ , and

$y(x; \varepsilon) \approx x + 1 - 2e^{(x-1)/\varepsilon}$ ; in this case the boundary layer would be next to  $x = 1$  instead!

Here's how we can think about this problem using the method of dominant balance. In the boundary layer, the dominant balance in the ODE is  $A$  (the highest derivative, proportional to  $\varepsilon$ ) vs.  $B$  (the highest derivative in the unperturbed ODE). This yields an approximate 'inner' solution on which we can apply the  $x = 0$  BC:

$$\text{Inner problem } (x = O(\varepsilon)): \quad \varepsilon y'' + y' = 0, \quad y(0) = 1 \Rightarrow y_{inner}(x) = 1 - c_1 + c_1 e^{-x/\varepsilon} \quad (9.2)$$

Outside the boundary layer, the dominant balance is  $B$  vs.  $C$ . This gives the unperturbed ODE, and the  $x = 1$  BC is in this region:

$$\text{Outer problem } (x \gg O(\varepsilon)): \quad y' = 1, \quad y(1) = 0 \Rightarrow y_{outer}(x) = x - 1 \quad (9.3)$$

For the inner solution (9.2) to match the outer solution (9.3) across the edge of the boundary layer,

$$\lim_{x/\varepsilon \rightarrow \infty} y_{inner}(x) = \lim_{x \rightarrow 0} y_{outer}(x)$$

$$1 - c_1 = 0 - 1 \Rightarrow c_1 = 2$$

In this way, we have recovered the approximate solution (9.1) to the exact problem. This solution is asymptotically accurate as  $\varepsilon \rightarrow 0$  in both the ‘outer’ sense (with  $x$  fixed, in which case the solution asymptotes to the outer solution), and the ‘inner’ sense (with  $x/\varepsilon$  fixed, in which case the solution asymptotes to the inner solution). The *uniformly valid* approximation  $y_{unif}(x; \varepsilon) = x - 1 + 2e^{-x/\varepsilon}$  of (9.1) encompasses both the inner and outer solutions in their respective regimes of validity. The inner, outer and exact solutions are plotted in Fig. 9.1 for  $\varepsilon = 0.1$ .

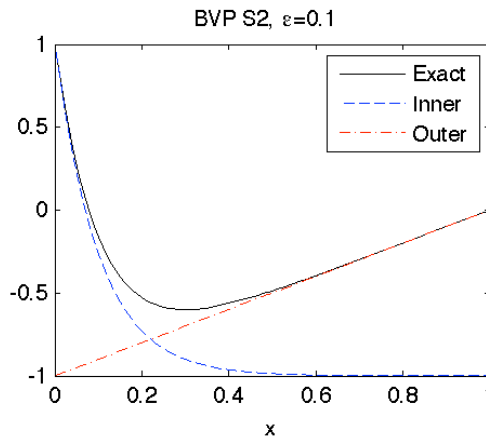


Fig 9.1. Asymptotic solution of (S2) showing boundary layer next to  $x = 0$ .

Example S3: Approximately solve the singularly perturbed BVP

$$\underbrace{-\varepsilon y''}_A + \underbrace{2y'}_B + \underbrace{y^2}_C = 0, \quad \underbrace{y(0) = 1}_{BC0}, \quad \underbrace{y(1) = 0}_{BC1}, \quad 0 < \varepsilon \ll 1 \tag{S3}$$

This BVP can be derived as the steady state concentration of a chemical with concentration  $y(x)$  with nondimensional horizontal diffusivity  $\varepsilon$  flowing in a tube from  $x = 0$ , where its concentration is 1, to  $x = 1$  with speed 2, while reacting with itself at a rate  $y^2$ . At  $x = 1$ , the chemical is totally removed from the tube so  $y(1) = 0$ .

One obvious new feature is the nonlinear term, which prevents the BVP from being solved exactly. A more subtle change from (S2) is the minus sign on the  $\varepsilon$ . As with (S2) we follow the

*Method for singularly perturbed ODEs with boundary layers:*

- (1) Look for an outer solution with dominant balance  $B$  vs.  $C$
- (2) Look for an inner solution with dominant balance  $A$  vs.  $B$ , applying relevant BCs.
- (3) Asymptotically match the solutions and apply remaining BCs.

Outer (unperturbed) problem:

$$2y' + y^2 = 0 \Rightarrow -2 \frac{dy}{y^2} = dx \Rightarrow \frac{2}{y} = x - x_0 \Rightarrow y_{outer}(x) = \frac{2}{x - x_0}$$

By appropriate choice of the constant of integration  $x_0$ , this solution can be made to satisfy one BC but not both, and it is not yet obvious which BC to impose. This dominant balance of  $B$  vs.  $C$  is consistent, since  $A = -\varepsilon y''_{outer} = O(\varepsilon) \ll B, C = O(1)$ .

Inner problem:

For  $A$  to be as large as either of  $B$  or  $C$ ,  $y$  must vary by a large fraction of itself over a distance  $L \ll 1$  (which we must determine). The outer solution and its mismatch with the BCs are both  $O(1)$ , so we assume that  $y$  is  $O(1)$  and scale all three terms. In this case,

$$A = \varepsilon y'' = O(\varepsilon / L^2), \quad B = 2y' = O(1 / L), \quad C = y^2 = O(1)$$

Thus  $B \gg C$ , so the dominant balance must be  $A$  vs.  $B$ , which implies the scaling  $L = \varepsilon$  and the inner problem

$$-\varepsilon y'' + 2y' = 0 \Rightarrow y_{inner}(x) = c_0 + c_1 e^{2x/\varepsilon}$$

The inner solution increases exponentially with lengthscale  $L = \varepsilon$  to the right, so there is no way to use it to match the left BC0 at  $x = 0$  to the more slowly varying outer solution. However, it can be used to match the right BC1 at  $x = 1$  by setting

$$0 = y_{inner}(1) = c_0 + c_1 e^{2/\varepsilon} \Rightarrow c_1 e^{2/\varepsilon} = -c_0 \Rightarrow y_{inner}(x) = c_0 \{1 - e^{2(x-1)/\varepsilon}\}, \quad 1 - x = O(\varepsilon)$$

This represents a boundary layer adjacent to  $x = 1$ , like the  $\varepsilon < 0$  case of (S2).

Matching outer vs. inner and applying BC0:

The outer solution must be valid for  $0 \leq x < 1 - O(\varepsilon)$ , in particular at  $x = 0$ . Hence we apply BC0:

$$1 = y_{outer}(0) = \frac{2}{0 - x_0} \Rightarrow x_0 = -2 \Rightarrow y_{outer}(x) = \frac{2}{x + 2}$$

Lastly, we must match with the inner solution:

$$\lim_{(1-x)/\varepsilon \rightarrow \infty} y_{inner}(x) = \lim_{x \rightarrow 1} y_{outer}(x) \Rightarrow c_0 = 2/3$$

We conclude that

$$y(x; \varepsilon) \sim \begin{cases} \frac{2}{3} \{1 - e^{2(x-1)/\varepsilon}\}, & 0 \leq 1-x \ll 1 \quad \text{'inner' solution} \\ \frac{2}{x+2} & 1-x \gg \varepsilon \quad \text{'outer' solution} \end{cases} \quad \text{for } \varepsilon \ll 1.$$

Lastly, by regarding the inner solution as a correction term to the outer solution within the boundary layer  $1-x = O(\varepsilon)$ , we can combine these two solution regimes into a uniformly valid solution across all  $x$ , both inside and outside the boundary layer:

$$y(x) \sim y_{unif}(x) = y_{outer}(x) + y_{inner}(x) - y_{outer}(1) = \frac{2}{x+2} - \frac{2}{3} e^{2(x-1)/\varepsilon}, \quad \varepsilon \ll 1$$

Fig. 9.2 shows the inner, outer and uniformly valid asymptotic approximations to the solution for  $\varepsilon = 0.1$ , clearly showing the boundary layer structure and how the uniformly valid solution captures both solution regimes.

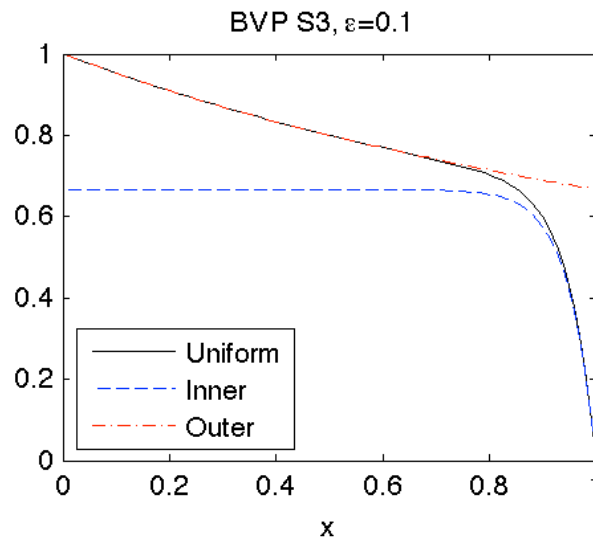


Fig. 9.2: Asymptotic solution of (S3) showing boundary layer next to  $x = 1$ .