Lecture 18: Multiresolution Wavelet Analysis

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Refs: Matlab Wavelet Toolbox help.

18.1 Interpretation of Haar wavelets in terms of filters

We can interpret the level-1 Haar detail and average vectors as filtering with the weight vector

\[ \mathbf{w}^1 = [w_0, w_1, \ldots, w_{-1}] = 2^{-1/2}[1, -1, 0, 0, \ldots, 0] \]
\[ \mathbf{v}^1 = [v_0, v_1, \ldots, v_{-1}] = 2^{-1/2}[1, 1, 0, 0, \ldots, 0] \] (18.1.1)

and then keeping only every second component of the filtered field. Their spectral responses are simply obtained from the Z-transform

\[ R_w(\omega) = 2^{-1/2} (1 - Z) \bigg|_{Z = \exp(i \omega \Delta t)} = 2^{1/2} i e^{i \omega \Delta t/2} \sin(\omega \Delta t/2) \]
\[ R_v(\omega) = 2^{-1/2} (1 + Z) = 2^{1/2} e^{i \omega \Delta t/2} \cos(\omega \Delta t/2) \] (18.1.2)

The spectral power of each of these filters is seen in Figure 1a. The level-1 detail filter is a coarse high-pass filter; the level-1 average filter is a coarse low-pass filter.

18.2 Extension to multiresolution analysis

If we now apply the Haar transform to the level-1 average vector \( \mathbf{a}^1 \) of length \( N/2 \), we can write

\[ d_m^2 = 2^{-1/2} (a_{2m-1} - a_{2m}) = 2^{-1} (u_{4m-3} + u_{4m-2} - u_{4m-1} - u_{4m}) \]
\[ a_m^2 = 2^{-1/2} (a_{2m-1} + a_{2m}) = 2^{-1} (u_{4m-3} + u_{4m-2} + u_{4m-1} + u_{4m}) \] (18.2.1)

Again, we can write this in terms of level 2 wavelet vectors. We can write

\[ d_m^2 = W_m^2 \cdot \mathbf{u} \]
\[ a_m^2 = V_m^2 \cdot \mathbf{u} \] (18.2.2)
where the \textbf{level-2 wavelets} are
\[
W_1^2 = 2^{-1}[1, 1, -1, 0, 0, 0, 0, \ldots, 0, 0] \\
W_2^2 = 2^{-1}[0, 0, 0, 1, 1, -1, 0, \ldots, 0, 0] \\
\vdots \\
W_{N/4}^2 = 2^{-1}[0, 0, 0, 0, \ldots, 1, 1, -1, -1] \tag{18.2.3}
\]
and the level-2 scaling signals are
\[
V_1^2 = 2^{-1}[1, 1, 1, 0, 0, 0, 0, \ldots, 0, 0] \\
V_2^2 = 2^{-1}[0, 0, 0, 1, 1, 1, 0, \ldots, 0, 0] \\
\vdots \\
V_{N/4}^2 = 2^{-1}[0, 0, 0, 0, \ldots, 1, 1, 1, 1] \tag{18.2.4}
\]
Again, these vectors are all mutually orthogonal and also orthogonal to the level-1 wavelets. The extension to higher levels is clear.

The inverse of this 2-level Haar transform can be expressed
\[
u = \underbrace{A^2 + D^2}_{A_1} + D^1. \tag{18.2.5}
\]

where
\[
A^2 = 2^{-1}(a_1^2, a_1^2, a_1^2, \ldots, a_{N/4}^2, a_{N/4}^2, a_{N/4}^2, a_{N/4}^2) = \sum_{m=1}^{N/2} d_m^2 V_m^2 \tag{18.2.6}
\]
and similarly
\[
D^2 = \sum_{m=1}^{N/2} d_m^2 W_m^2 \tag{18.2.7}
\]
Figure 2: The wavelets and the scaling vector for a level-3 analysis of a time series of length 8.
The extension to a $P$-level Haar transform should now be clear.

Again, the levels $p = 1, \ldots, P$ of the multilevel Haar transform can be regarded as filtering followed by binary subsampling. Fig. 1b shows the spectral power of the equivalent detail filters for each level, and for the average filter at the maximum level $P$, for the case $P = 3$. We can interpret the level-$p$ detail filter as being an approximate bandpass filter centered on a frequency corresponding to a period $2^p \Delta t$, consistent with the level-$p$ wavelets extracting the information in the time series on this timescale.

### 18.3 Multiresolution wavelet analysis in Matlab

The Matlab wavelet toolbox has an extensive set of functions for single-level and multiresolution wavelet analysis. For those of you who do not have access to this toolbox, I have written Matlab functions `dwtHaar` and `idwtHaar` (class web page) that do forward and inverse Haar transforms of arbitrary level; given the scaling coefficients of level $P$ and the wavelet coefficients up to level $P$, the inverse transform returns the contributions of the different levels (timescales) that add up to the original time series.

Wavelet toolbox functions can be used for multiresolution wavelet analysis. To sidestep the toolbox, I have written Matlab functions `dwtHaar` and `idwtHaar` (class web page) that do forward and inverse Haar transforms of arbitrary level; given the scaling coefficients of level $P$ and the wavelet coefficients up to level $P$, the inverse transform returns the contributions of the different levels (timescales) that add up to the original time series. A second part of `wavelet_jeleccum_notoolbox` illustrates a level-3 Haar wavelet analysis on the time series, using the `dwtHaar` and `idwtHaar` functions. Wavelet toolbox functions also can