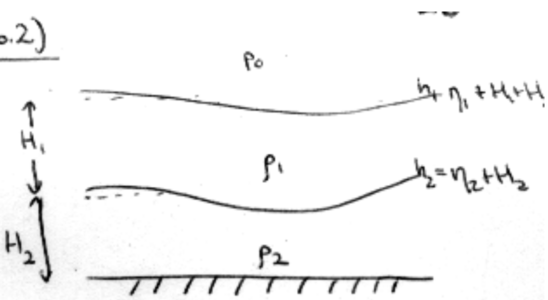
LSWE and Two layer flow (Gill 6.2)



Two superposed incompressible shallow fluid layers, f plane, no bottom topography.
Also: Small rel. density diff $\boxed{\frac{\rho_2-\rho_1}{\rho_2} \ll 1}$
Hydrostatic approx

$$p_1(x,y,z,t) = p_0 + \rho_1 g(h_1 - z)$$

$$p_2(x,y,z,t) = p_0 + \rho_1 g(h_1 - h_2) + \rho_2 g(h_2 - z)$$

$$\Rightarrow \frac{\partial p_1}{\partial x} = -\rho_1 g \frac{\partial h_1}{\partial x} = -\rho_1 g \frac{\partial \eta_1}{\partial x}$$

$$\frac{\partial p_2}{\partial x} = -\rho_1 g \frac{\partial}{\partial x}(h_1 - h_2) - \rho_2 g \frac{\partial h_2}{\partial x} = -\rho_1 g \frac{\partial \eta_1}{\partial x} - (\rho_2 - \rho_1) g \frac{\partial \eta_2}{\partial x}.$$

Mass continuity

$$\frac{\partial}{\partial t}(\underbrace{h_1 - h_2}_{\eta_1 - \eta_2 + H_1}) + H_1\left\{\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right\} = 0$$

$$\frac{\partial \underbrace{h_2}_{\eta_2 + H_2}}{\partial t} + H_2\left\{\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right\} = 0$$

... coupled pairs of PDEs. It is fruitful in this, like many other PDE problems to look for separable solutions, which in this context means η_1, η_2 have same horizontal/time structure:

$$\eta_2(x,y,t) = \mu \eta_1(x,y,t) \quad , \mu \text{ TBD.}$$

$$\Rightarrow \frac{\partial u_1}{\partial t} - f v_1 = - g \frac{\partial \eta_1}{\partial x} \qquad \text{similar for } \frac{\partial v_1}{\partial t}$$

$$\frac{\partial u_2}{\partial t} + f v_2 = -\frac{\rho_1}{\rho_2} g \frac{\partial \eta_1}{\partial x} - \frac{\rho_2 - \rho_1}{\rho_2} g \frac{\partial \eta_2}{\partial x} = -\nu g \frac{\partial \eta_1}{\partial x} \quad , \nu = \frac{\rho_1}{\rho_2} + \frac{\rho_2 - \rho_1}{\rho_2}\cdot\mu$$

so $u_2 = \nu u_1$, $v_2 = \nu v_1$. Now sub into continuity eqns.

$$(1-\mu)\frac{\partial \eta_1}{\partial t} + H_1 \nabla\cdot\vec{u}_1 = 0$$

$$\mu \frac{\partial \eta_1}{\partial t} + \nu H_2 \nabla\cdot\vec{u}_1 = 0$$

$$\Rightarrow \frac{1-\mu}{\mu} = \frac{H_1}{\nu H_2} \quad \text{for consistency. } (*)$$

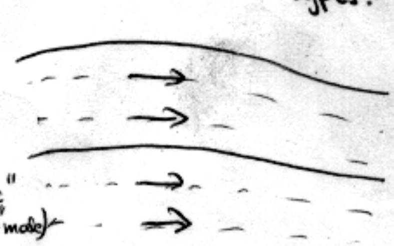
gives SWE with $H(\mu) = \dfrac{H_1}{1-\mu}$ for η_1, u_1 (the "equivalent depth")

Cross-multiply $(*)$ to get quadratic for μ. Two roots μ_0, μ_1 give two superposable modes types:

with $\mu_0 = O(1) \Rightarrow \nu_0 = \frac{\rho_1}{\rho_2} + \frac{\rho_2 - \rho_1}{\rho_2}\cdot\mu_0 \approx 1 \Rightarrow$

$$\frac{1-\mu_0}{\mu_0} \approx \frac{H_1}{H_2} \qquad \mu_0 = \frac{H_2}{H_1 + H_2}$$

$$H_0 = H_1 + H_2$$

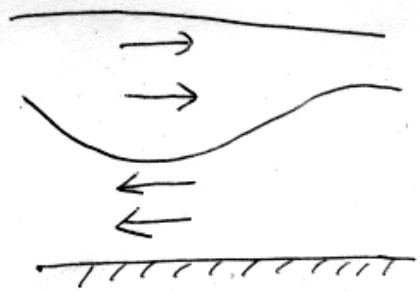


The two fluid layers are moving together as one with almost no impact of the density interface ("barotropic" or "external" mode).
Other mode has $\mu_1 \gg 1 \Rightarrow$

$$\frac{1-\mu_1}{\mu_1} \approx -1 = \frac{H_1}{\nu H_2} \quad \Rightarrow \nu_1 \approx -\frac{H_1}{H_2} = \underbrace{\frac{\rho_1}{\rho_2}}_{\approx 1} + \frac{\rho_2 - \rho_1}{\rho_2}\mu_1 \quad \Rightarrow \frac{u_2}{u_1} = -\frac{H_1}{H_2},$$

$$\frac{\rho_2-\rho_1}{\rho_2}\cdot\mu_1 = -\frac{H_1+H_2}{H_2}$$. Thus $\eta_1 \ll \eta_2$ (upper interface nearly flat)

$H_1 u_1 \approx -H_2 u_2$ (compensating to make no net mass flux)



... the "internal" or "baroclinic mode" with

$$H_{e1} \approx \frac{H_1}{1-\mu_1} \approx \frac{H_1 H_2}{H_1+H_2}\cdot\frac{\rho_2-\rho_1}{\rho_2}$$

The shallow water wave speed is $c_1 = \sqrt{gH_{e1}} = \sqrt{g' H_{eff}}$ where $g' = g\frac{\rho_2-\rho_1}{\rho_2}$ is the effective or reduced gravity and $\boxed{H_{eff} = \frac{H_1 H_2}{H_1+H_2}}$.

<u>To summarize</u> :

	<u>Barotropic mode type</u>	<u>Baroclinic mode type</u> $\left(\frac{\rho_2-\rho_1}{\rho_2} \ll 1\right)$
Equiv depth	$H_e \approx H_1+H_2 = H$	$H_e = \frac{H_1 H_2}{H_1+H_2}$
Eff gravity	$g' \approx g$	$g' = g\frac{\rho_2-\rho_1}{\rho_2} \ll g$
Wave speed	$c_0^2 \approx gH$	$c_0^2 = g' H_e$
Motions	$u_1 \approx u_2$	$H_1 u_1 \approx -H_2 u_2$

<u>Rough numbers</u>

	H_1	H_2	$\frac{\rho_2-\rho_1}{\rho_2}$	Barotropic c_0	Barotropic $R=\frac{c_0}{f}$	Baroclinic g'	Baroclinic H_e	Baroclinic c_0	Baroclinic $R=\frac{c_0}{f}$
Atm	5 km	5 km	10^{-1}	300 $\frac{m}{s}$	3000 km	1 $\frac{m}{s^2}$	2.5 km	50 $\frac{m}{s}$	500 km
Ocn	1 km	4 km	10^{-3}	200 $\frac{m}{s}$	2000 km	10^{-2} $\frac{m}{s^2}$	0.8 km	3 $\frac{m}{s}$	30 km

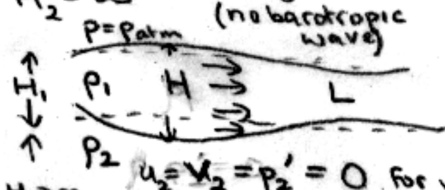
Each mode type supports both Poincare waves + steady geostrophic modes.

<u>Limits</u> :

$1\frac{1}{2}$ layer approx to ocean thermocline : H_1 finite, $H_2 = \infty$ $\Rightarrow$ Barotropic $c_0 = \infty$ (no barotropic wave); Baroclinic $c_0 = \sqrt{g' H_1}$, $u_2 = 0$

∴ no motion in lower layer $\Rightarrow p_2' = 0 \Rightarrow$

$$\frac{\eta_1}{\eta_2} \approx -\frac{\rho_2-\rho_1}{\rho_1}$$



$u_2 = v_2 = p_2' = 0$ for waves.

∴ only one (baroclinic) Poincare wave mode class.

(can still have <u>steady</u> barotropic geostrophic flows with $\vec{u}_1(x) \approx \vec{u}_2(x)$).

Baroclinic Poincare Waves in Lake Michigan (from Gill)
... close to a $1\frac{1}{2}$ layer flow.

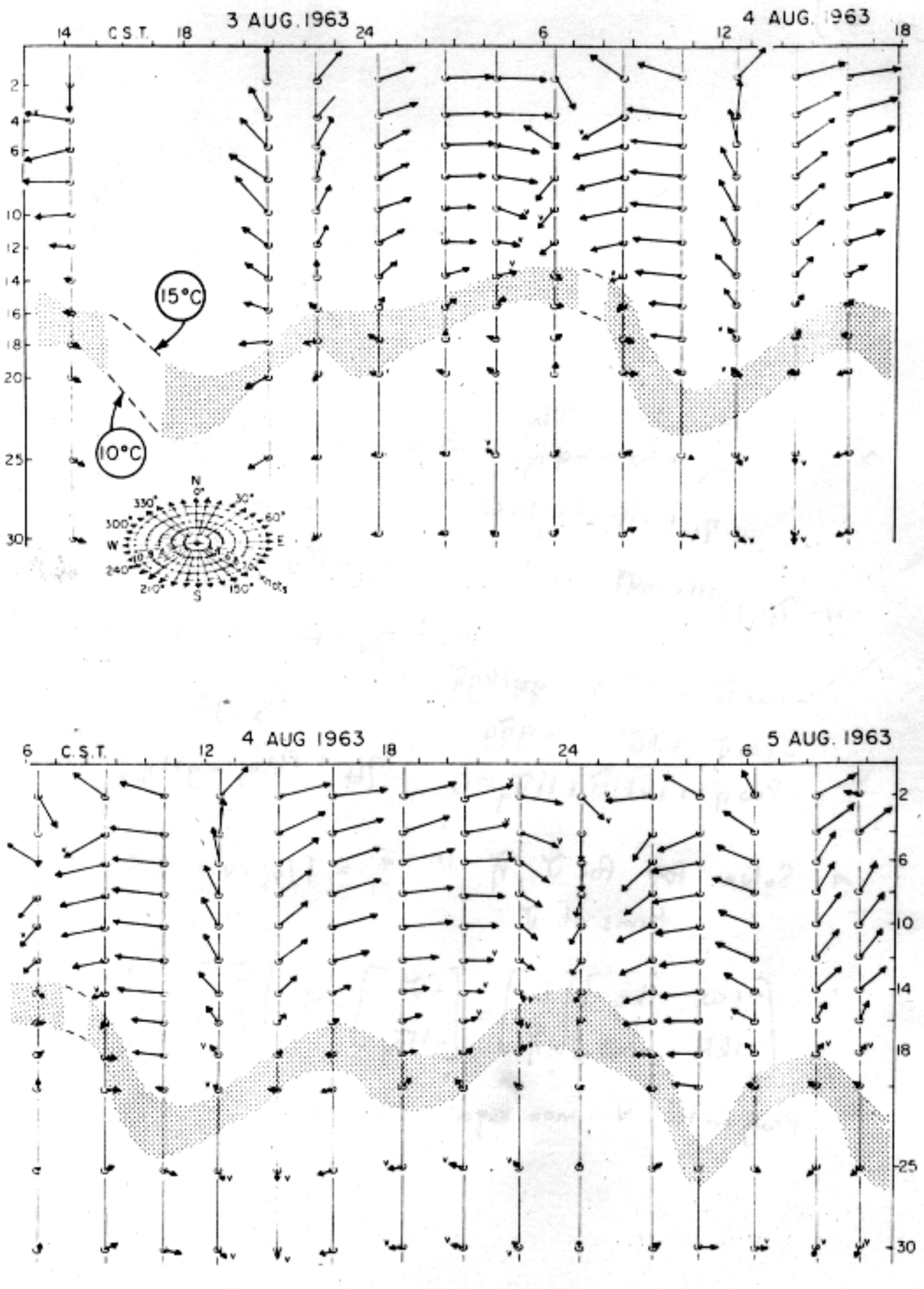
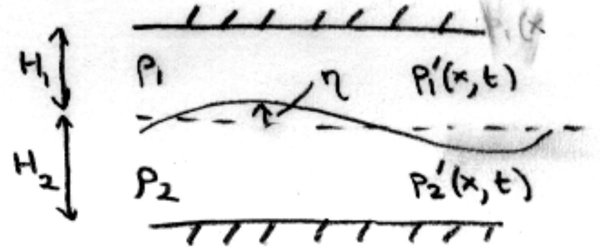


Fig. 8.2. Observations of currents and thermal structure in the upper 30 m of Lake Michigan, August 3–5, 1963. The arrows indicate direction and speed in accordance with the current rose, and the shaded portion indicates the thermocline as denoted by the 10° and 15° isotherms. The two diagrams, which are overlapping in time, are separated by 17 hr, which is the dominant period. The local inertial period is 17.5 hr. Note the approximate two-layer structure in both temperature and velocity, and the anticyclonic rotation with time of velocity vectors. [From Mortimer (1971, Fig. 85).]

"Rigid lid" approximation (ignore y-variation for simplicity)

No free surface; wall above upper fluid



Mom

$$\frac{\partial u_1}{\partial t} - f v_1 = -\frac{1}{\rho_1}\frac{\partial p_1'}{\partial x}$$

$$\frac{\partial u_2}{\partial t} - f v_2 = -\frac{1}{\rho_2}\frac{\partial p_2'}{\partial x}$$

Hydrostatic approx

$$p_2' = p_1' + (\rho_2 - \rho_1) g \eta$$

Note rigid upper lid can support horizontal pressure perturbations $p_1'(x,t)$; p_1' must be solved for.

Mass

$$\frac{\partial}{\partial t}(H_1 - \eta) + H_1 \frac{\partial u_1}{\partial x} = 0 \quad \text{(add)}$$

$$\frac{\partial}{\partial t}(H_2 + \eta) + H_2 \frac{\partial u_2}{\partial x} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(H_1 u_1 + H_2 u_2\right) = 0$$

$\Rightarrow$ Flow is sum of a uniform flow $u_1 = u_2 = \text{const.}$ (which we can zero out by choosing our reference frame to be moving with the uniform flow) + a flow with

$$M = H_1 u_1 + H_2 u_2 = 0 \quad \Rightarrow \text{no net vertically-integrated volume flux in any column}$$

$$\Rightarrow \quad u_2 \approx -\frac{H_1}{H_2} u_1 \quad \text{(i.e. all modes with } u \neq 0 \text{ are baroclinic)}$$

$$\Rightarrow p_2' = -\frac{H_1}{H_2} p_1' \quad \Rightarrow \quad p_1' + (\rho_2 - \rho_1) g \eta = -\frac{H_1}{H_2} p_1'$$

so $$p_1' = -\frac{(\rho_2 - \rho_1) g \eta}{1 + H_1/H_2}$$; this is the same pressure perturbation we'd see in the baroclinic mode type with an upper free surface.

Conclusion: Rigid lid supports the same baroclinic modes as a 2-layer fluid with upper free surface, but does not support barotropic wave modes. Barotropic geostrophic modes are still possible.