LSWE Energy Equation (G 11 5.7)

Energy is a fundamental quantity conserved in nondissipative physical systems. How does this play out for LSWE.

Take $u \times$ (x-mom eqn) + $v \times$ (y-mom eqn):

$$u\left(\frac{\partial u}{\partial t} - fv\right) + v\left(\frac{\partial v}{\partial t} + fu\right) = -g\left\{u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right\}$$

Since coriolis force $(-fv, fu) \perp (u,v)$, it does no mechanical work and cancels out of energy eqn. Also note mass cons.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{H}\frac{\partial \eta}{\partial t} \quad \Rightarrow$$

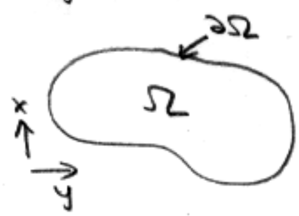
$$\frac{\partial}{\partial t}\left\{\frac{1}{2}(u^2+v^2)\right\} = -\frac{\partial}{\partial x}(ug\eta) - \frac{\partial}{\partial y}(vg\eta) + g\eta\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{\frac{1}{H}\frac{\partial \eta}{\partial t}}$$

Multiply by ρH

$$\frac{\partial}{\partial t}\left\{\underbrace{\rho\frac{H}{2}(u^2+v^2)}_{\text{Column-integrated KE}} + \underbrace{\rho g\frac{\eta^2}{2}}_{\text{Column-integrated available potential energy (APE)}}\right\} = \underbrace{-\nabla_h \cdot [\vec{u}_h \overbrace{\rho g \eta}^{p'} H]}_{\text{net pressure work done by each column on its neighbors}}$$

If we integrate across a closed or periodic domain Ω,

$$-\int_\Omega \nabla_h \cdot [\vec{u}_h p' H]\, dA = -\int_{\partial\Omega} (\hat{n}\cdot\vec{u}_h)\, p' H\, dA$$



$$= 0$$ because no normal flow at walls, periodic cancellation at periodic boundaries

$$\Rightarrow \frac{\partial}{\partial t}\int_\Omega \left\{\rho\frac{H}{2}(u^2+v^2) + \frac{\rho g \eta^2}{2}\right\} dA = 0 \quad \text{(Energy conservation)}.$$

In nonrotating gravity waves, energy is <u>equipartitioned</u> between KE, APE, but for Poincare waves, KE > APE, particularly as $\omega \to f$ (as in inertial oscillations with $\eta = 0$, for which APE = 0 but KE > 0). For geostrophic modes of wavenumber $\kappa = |\vec{k}|$, $\frac{KE}{APE} = k^2R^2$, R = Rossby rad. so mostly KE for short wavelengths, PE for long wavelengths.