Rossby waves (CR6.4)

A very important class of waves — requires a gradient of mean potential vorticity $\frac{f}{H}$, due either to spatial variations in f, or to variations in depth.

To examine effect of variations in $f(y)$, use the β-plane approx. The LSWE become

$$\frac{\partial u}{\partial t} - (\overbrace{f_0+\beta y}^{f})v = -g\frac{\partial \eta}{\partial x} \qquad (1)$$

$$\frac{\partial v}{\partial t} + (f_0+\beta y)u = -g\frac{\partial \eta}{\partial y} \qquad (2)$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0. \qquad (3)$$

Note that steady motion is no longer possible since $\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial y} = \left[-\frac{d}{dy}\left(\frac{g}{f_0+\beta y}\right)\right]\cdot\frac{\partial \eta}{\partial x}$

$\neq 0$ if η varies w. x! $\Rightarrow \frac{\partial \eta}{\partial t}\neq 0$

If we let $\begin{bmatrix} u \\ v \\ h\end{bmatrix} = \mathrm{Re}\begin{bmatrix}\tilde u(y)\\ \tilde v(y)\\ \tilde h(y)\end{bmatrix}e^{i(kx-\omega t)}$ and eliminate $a, \tilde\eta$, then from (1), (3)

$$\begin{bmatrix}\tilde u\\ \tilde\eta\end{bmatrix} = \frac{1}{\omega^2-gHk^2}\begin{bmatrix} i\omega f\tilde v - ikgH\frac{d\tilde v}{dy}\\ ikfH\tilde v - i\omega H\frac{d\tilde v}{dy}\end{bmatrix} \quad \text{just as before}$$

Plug into (2):

$$-i\omega\left[\left\{\underset{(Ia)}{\omega^2}-(\underset{(Ib)}{f^2}+\underset{(Ic)}{gHk^2})\right\}\tilde v + \underset{(Id)}{gH\frac{d^2\tilde v}{dy^2}}\right] + \underset{(II)}{ikgH\beta\tilde v} = 0 \qquad (4)$$

(II): component due to β-effect.

Consider a mid-latitude β-plane, and restrict to channel with $[\beta y]/f_0 \ll 1$ (as required for β-plane approx.). Also assume $kr_0 \gg 1$ (less than planetary scale). Then

$$f^2 \approx f_0^2$$

and for Poincaré waves with $\omega \geq f$,

$$\frac{[II]}{[Ic]} = \left[\frac{kgH\beta}{k^2gH\omega}\right] = \left[\frac{\beta}{\omega k}\right] \geq \left[\frac{\beta}{fk}\right] = \left[\frac{1}{kr_0}\right] \ll 1.$$

so Poincaré waves satisfy

$$\omega^2 \approx f_0^2 + gH(k^2+l^2) \quad \text{as before.}$$

But if $\omega/f_0 \ll 1$, then (II) can be important. Note the temporal Rossby number

$$Ro_T = \frac{[\partial v/\partial t]}{[fv]} = \frac{\omega}{f_0} \ll 1 \text{ for this mode}$$

so motions are nearly geostrophic:

$$u \approx u_g \approx -\frac{g}{f_0}\frac{\partial \eta}{\partial y}$$

$$v \approx v_g \approx \frac{g}{f_0}\frac{\partial \eta}{\partial x} \qquad \text{Polarization relations}$$

The eqn (4) for v becomes approximately,

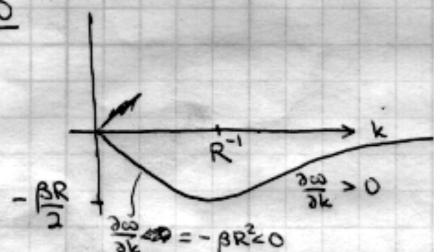
$$-i\omega\left[-(f_0^2+gHk^2)\tilde{v} + gH\frac{d^2\tilde{v}}{dy^2}\right] + ikgH\beta\tilde{v} = 0$$

Since this is const. coeff, let $\tilde{v} = \hat{v}e^{ily}$. Then

$$\omega = \frac{-\beta k}{k^2+l^2+R^{-2}} \qquad \text{(Rossby wave dispersion).}$$

Note CR uses l, m not k, l but this isn't standard!

$\underline{l=0}$



Note all waves have $c_{ph} < 0$ (Westerly phase propagation) but if $kR > 1$, $c_g > 0$.

Also note

$$[|\omega_{max}|] = \left[\frac{\beta R}{2}\right] \sim \frac{R}{a}\cdot f$$

so as long as $R \ll a$, $[\omega]/f \ll 1$ in accordance with assumptions

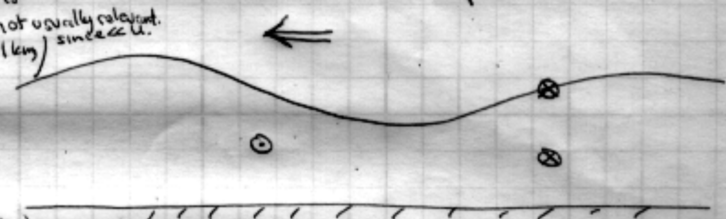
Note long wave speed ($\beta = 1.4\times10^{-11}\,m^{-1}s^{-1}$) is

$$\beta R^2 = \begin{cases} 2\,\frac{m}{s} & \text{atm. (internal R.W.) - not usually relevant since} \ll U. \\ 1.5\,\frac{cm}{s} & \text{ocean } (g'=0.01,\ H=1\,km),\ R = 30\,km \end{cases}$$

In atm, mainly barotropic waves in which $g = 10\,\frac{m}{s}$, $H = 10\,km$, $R = 3000\,km$

$\Rightarrow \beta R^2 = 120\,\frac{m}{s}$. For waves with $k = \frac{2\pi}{10000\,km}$

$$\frac{\omega}{k} = -\frac{\beta}{k^2+R^{-2}} \approx -\frac{\beta}{k^2} = -35\,\frac{m}{s} \approx -U \Rightarrow \text{standing in jet.}$$



Vorticity Dynamics - Why do Rossby waves propagate? L14

Vorticity eqn: $\frac{\partial}{\partial x}\left\{\frac{\partial v}{\partial t} + fu + g\frac{\partial h}{\partial y}\right\} - \frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x}\right\} = 0.$

$$\Rightarrow \frac{\partial\zeta}{\partial t} + f\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{-\frac{1}{H}\frac{\partial h}{\partial t}} + \beta v = 0.$$

$$\Rightarrow \frac{\partial}{\partial t}\left\{\zeta - \frac{f_0}{H}h\right\} + \underbrace{\frac{df}{dy}}_{\beta}\cdot v = 0$$

$$\left(\frac{Dq}{Dt}\right)_{\text{linearized}} = \frac{\partial q'}{\partial t} + v\frac{\partial \bar{q}}{\partial y} = 0, \quad q' = \left(\zeta - \frac{f}{H}h\right)/H, \quad \bar{q} = \frac{f_0+\beta y}{H} = \frac{f(y)}{H}$$

$q = \bar{q} + q'$ = linearization of $\frac{\zeta + f(y)}{H+h}$,

so our vorticity eqn once again reduces to

$$PV = \frac{\text{absolute vorticity}}{\text{layer depth}} \text{ is conserved.}$$

Furthermore, since $\omega \ll f$ the motions must nearly be geostrophic, so

$$-\cancel{\frac{\partial v}{\partial t}} - fv = -g\frac{\partial h}{\partial x} \qquad u \approx u_g = -\frac{g}{f}\frac{\partial h}{\partial y},\ v \approx v_g = \frac{g}{f}\frac{\partial h}{\partial x}$$

$$-\cancel{\frac{\partial u}{\partial t}} + fu = -g\frac{\partial h}{\partial y} \Rightarrow \zeta \approx \zeta_g = \frac{g}{f}\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) \text{ "quasigeostrophic approx"}$$

$$q' \approx q'_g = \frac{g}{fH}\left\{\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} - \frac{h}{R^2}\right\},\ R = \sqrt{gH}/f$$

Thus:

$$\frac{\partial}{\partial t}q'_g + v_g\frac{\partial\bar{q}}{\partial y} \approx 0 \quad \text{for low freq motions } (\omega \ll f)$$

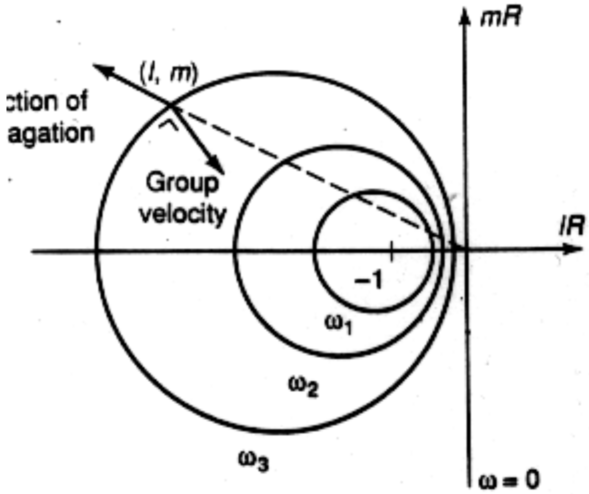


Figure 6-5 Geometric representation of the planetary-wave dispersion relation. Each circle corresponds to a single frequency, with frequency increasing with decreasing radius. The group velocity of the (*l*, *m*) wave is a vector perpendicular to the circle at point (*l*, *m*) and directed toward its center.

Standing Barotropic Rossby Waves in Atm. Jet Stream

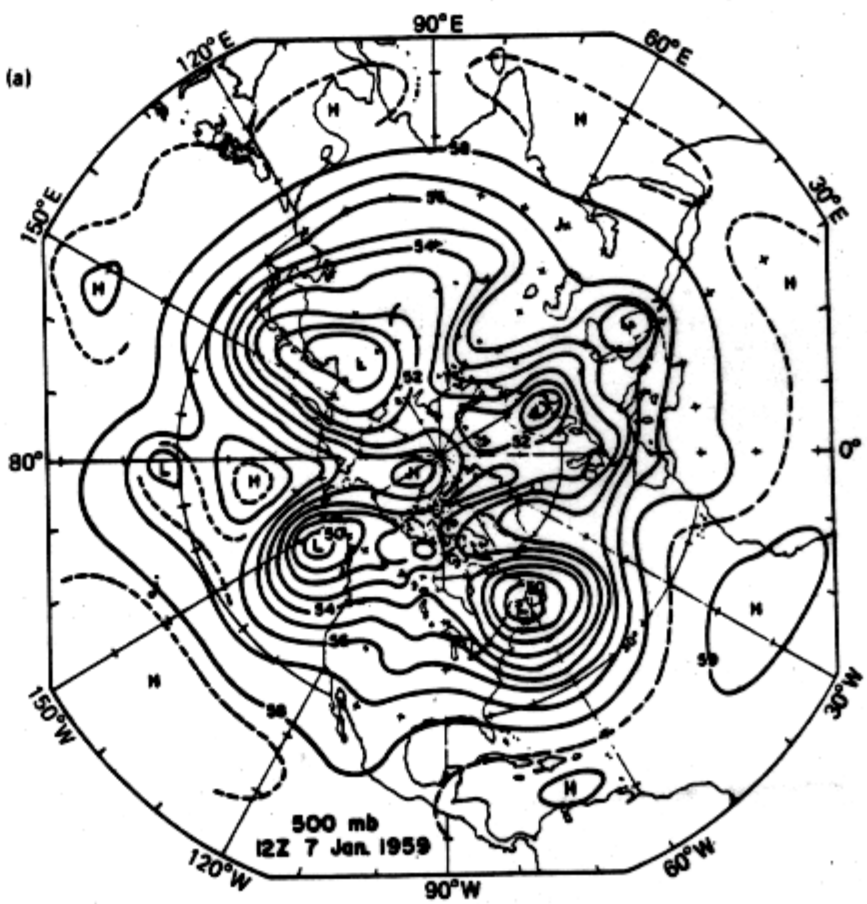


Figure 1.2.1(a) Isolines of constant pressure (isobars) at a level which is above roughly one-half the atmosphere's mass. The isobars very nearly mark the streamlines of the flow (Palmén and Newton, 1969).

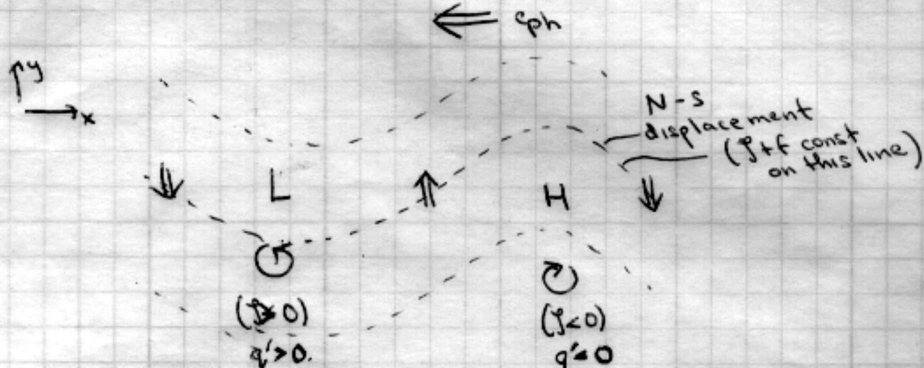
i.e. $\frac{f}{H}\frac{\partial}{\partial t}\left\{\frac{g}{f}\left[\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right]-\frac{f}{H}h\right\} + \frac{g}{f}\frac{\partial h}{\partial x}\cdot\frac{\beta}{H} = 0.$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\left[\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}-\frac{h}{R^2}\right]+\beta\frac{\partial h}{\partial x}=0}$$ QG PV eqn

$\Rightarrow$ If $h = \mathrm{Re}\,\hat{h}e^{i(kx+ly-\omega t)}$,

$$-i\omega\left[-k^2-l^2-\frac{1}{R^2}\right]+ik\beta=0 \Rightarrow \omega=\frac{-\beta k}{k^2+l^2+R^{-2}}$$ (as before)

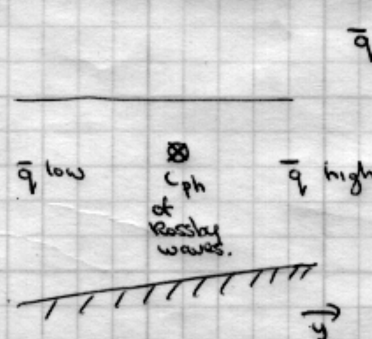
High absolute vorticity air is brought down to W of low, causing the vort. max to move W-ward.



Notes: Same effect of varying $\bar{q}$ can be got by varying $H = H(y)$ in place of β:

$\Rightarrow$ Topographic Rossby waves. $\beta_{eff} = H\frac{\partial\bar{q}}{\partial y} = H\frac{\partial}{\partial y}\left(\frac{f}{H}\right) = -\frac{f}{H}\frac{dH}{dy}.$

This β_{eff} can be quite a bit larger than β, producing faster topographic Rossby waves on, e.g., continental shelf margins.



SWE Quasigeostrophy (P3.12)

For low frequency motions with timescale $T \ll f^{-1}$,

$$u \approx u_g = -\frac{g}{f}\frac{\partial h}{\partial y} \Rightarrow \zeta_g = \frac{g}{f}\nabla^2 h, \quad \frac{D}{Dt} \approx \frac{D_g}{Dt} = \frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}$$

$$v \approx v_g = \frac{g}{f}\frac{\partial h}{\partial x}$$

$$q \approx q_g = \frac{f+\zeta_g}{h}, \text{ so } 0 = \frac{Dq}{Dt} \approx \frac{D_g q_g}{Dt} \quad \text{and} \quad f + \frac{g}{f}\nabla^2 h - q_g h = 0$$

Thus, knowing $q_g(x,y,t)$ at $t=0$, we deduce $h(x,y,0)$, u_g, v_g, hence can advect q_g.