Standing topographically-driven Poincare waves in a mean flow on an *f*-plane

Consider the LSWE on an f-plane for a fluid layer of depth H and eastward mean flow $U > 0$. Small y-independent disturbances are created by small bottom topography $z_b(x) << H$.

The time dependent LSWE for this flow are ($\partial/\partial y = 0$, linearized $D/Dt = \partial/\partial t + U\partial/\partial x$):

$$u_t + Uu_x - fv = -g\eta_x$$
$$v_t + Uv_x + fu = 0$$
$$\eta_t + U(\eta - z_b)_x + Hu_x = 0$$

With $z_b = 0$, these equations admit Doppler-shifted Poincare waves $\eta = \eta_0 \exp i(kx - \omega t)$ with

$$\underbrace{\tilde{\omega}}_{\text{'intrinsic' frequency}} = \omega - Uk = \pm\left(f^2 + c_0^2k^2\right)^{1/2}, \quad c_0 = (gH)^{1/2}.$$

Their group and phase velocity obey the normal formula, but with U added to both. Similarly, the 'stationary mode' now consists of geostrophically balanced vortical disturbances which advect to the east at speed U.

Consider the steady 'standing wave' response to an isolated ridge $z_b = b \exp(-|x|/a)$, $b << H << a$. Now the LSWE (with ($\partial/\partial t = 0$) are:

$$Uu_x - fv = -g\eta_x \qquad (1)$$
$$Uv_x = -fu \qquad (2)$$
$$\eta = z_b - Hu/U \qquad (3)$$

They can be combined into a single ODE for $v(x)$,

$$s^2 v_{xx} - v = -g\, z_{bx}/f = (bg/fa) \exp(-|x|/a) \operatorname{sgn}(x), \quad s^2 = (c_0^2 - U^2)/f^2$$

Note that although v_{xx} has a step discontinuity at the origin, v and v_x are continuous there. Also, the flow must have zero PV perturbation everywhere ($\zeta/f = \eta/H$) since fluid columns conserve their PV as they are advected in the mean flow from the undisturbed upstream conditions.

The BC as as $x \to \pm\infty$ is derived by considering what happens to an undistrubed flow if the obstacle is suddenly raised up at a time $t = 0$. In this case, Poincare waves radiate energy away from the obstacle. If the flow can support a standing free left-moving Poincare wave, this wave will develop away from the origin out to infinity in the direction of its group velocity. Otherwise $v \to 0$ far from the obstacle.

There are two cases:

(1) $U < c_0$ (slow mean flow; $s^2 > 0$, no standing Poincare waves)

For this case, the solution must go to zero far from the obstacle and has the form

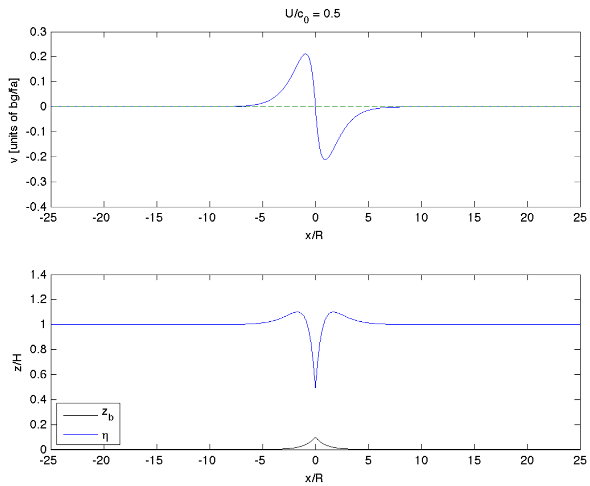
$$v(x) = \begin{cases} A_- \exp(x/s) + B_- \exp(x/a) & x < 0 \\ A_+ \exp(-x/s) + B_+ \exp(-x/a) & x > 0 \end{cases}$$

The second term is the inhomogeneous solution; the first term is the BC-obeying homogeneous solution. The coefficients $B_\pm$ are found by substitution, and $A_\pm$ by matching across $x = 0$:

$$B_\pm = \frac{\pm bg/fa}{(s/a)^2 - 1} = -A_\pm$$

The free surface height is most easily found from PV conservation: $\eta/H = \zeta/f = v_x/f$.

Matlab plots of v and η for $U = 0.5c_0$:



(2) $U > c_0$ (fast mean flow; standing Poincare waves supported)

For this case, set $k^2 = f^2/(U^2 - c_0^2)/$ (defines the standing Poincare wavenumber). For a left moving Poincare wave, $c_p < c_g$, so the standing wave, which has $c_p = 0$, must have $c_g > 0$, and

$$v(x) = \begin{cases} B_- \exp(x/a) & x < 0 \\ \underbrace{A_+ \exp(ikx) + A_- \exp(-ikx)}_{\text{standing Poincare waves fluxing energy rightward}} + B_+ \exp(-x/a) & x > 0 \end{cases}$$

Now $A_+ = A_- = -B_+$ Matlab plots for $U = 2c_0$ show the downstream standing Poincare wave: