Continuously Stratified Flow (CR 9½, G11 §.4)

Assume a reference state $\bar{p}(z)$, $\bar{\rho}_\theta(z)$, $\bar{\rho}(z)$, etc. in hydrostatic balance. and let $p' = p - \bar{p}$, etc. The Boussinesq equations of motion on an f or β plane are: (inviscid, adiabatic)

$$\frac{D\vec{u}}{Dt} + f\hat{k}\times\vec{u} = -\frac{1}{\rho_{00}}\nabla p' + b\hat{k} \quad , \; b = -\frac{g\rho_\theta'}{\rho_{00}} = \text{buoyancy}$$

$$\nabla\cdot\vec{u} = 0$$

$$\frac{D\rho_\theta}{Dt} = 0 \Rightarrow \frac{D\rho_\theta'}{Dt} + w\frac{d\bar{\rho}_\theta}{dz} = 0 \quad \text{or} \quad \frac{Db}{Dt} + \left(-\frac{g}{\rho_{00}}\frac{d\bar{\rho}_\theta}{dz}\right)w = 0$$

Note that the thermodynamics + dynamics interact thru buoyancy, which thus plays a pivotal role.

Stability

Consider displacing a fictional test parcel from z to $z+\delta z$. At this new height, what is its buoyancy?

$$b = -g\left\{\frac{\rho_\theta(\text{parcel}) - \bar{\rho}_\theta(z+\delta z)}{\rho_{00}}\right\} \quad , \; \rho_\theta(\text{parcel}) = \bar{\rho}_\theta(z)$$

$$= \frac{g}{\rho_{00}}\frac{d\bar{\rho}_\theta}{dz}\delta z \; < 0 \text{ if } \frac{d\bar{\rho}_\theta}{dz} < 0 \text{ (stably stratified)}$$

If it were to move as a result of buoyancy alone,

$$\frac{D^2}{Dt^2}\delta z = \frac{Dw}{Dt} = b \, .$$

Thus, defining the buoyancy frequency $N = \left(-\frac{g}{\rho_{00}}\frac{d\bar{\rho}_\theta}{dz}\right)^{\frac{1}{2}} \approx \left(-\frac{g}{\bar{\rho}_\theta}\frac{d\bar{\rho}_\theta}{dz}\right)^{\frac{1}{2}}$

we have $\qquad \approx \left(\frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dz}\right)^{\frac{1}{2}}$ (atm)

$$\delta z = \delta z_0 \cos Nt \, ,$$

parcel oscillates with frequency N.

Troposphere: $N \sim \left(\frac{10\,\frac{m}{s^2}}{300K}\,\frac{30\,K}{10^4 m}\right)^{\frac{1}{2}} = 10^{-2}s^{-1}$ $\quad$ ($N \sim 2\times10^{-2}s^{-1}$ in stratosphere)

Thermocline: $N \sim \left(\frac{10\,\frac{m}{s^2}}{1028\,\frac{kg}{m^3}}\cdot\frac{3\,\frac{kg}{m^3}}{1000\,m}\right)^{\frac{1}{2}} \sim 5\times10^{-3}s^{-1}$ $\quad$ ($N < 10^{-3}s^{-1}$ in deep ocean)

Interbia-Gravity Waves on an f-plane (CR 10)

Assume small-amplitude disturbances on a stationary reference state. Then

(1) $\qquad u_t - fv = -\frac{1}{\rho_{00}}p'_x$

(2) $\qquad v_t + fu = -\frac{1}{\rho_{00}}p'_y$

(3) $\qquad \mu w_t = -\frac{1}{\rho_{00}}p'_z + b \qquad \mu = \begin{cases} 1 & \text{full eqns} \\ 0 & \text{hydrostatic approx} \end{cases}$

(4) $\qquad u_x + v_y + w_z = 0$

(5) $\qquad \frac{\partial b}{\partial t} + N^2 w = 0$

If N^2 is constant in z, expect solutions $\begin{bmatrix} u \\ v \\ w \\ b \\ p' \end{bmatrix} = \mathrm{Re}\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{b} \\ \hat{p}' \end{bmatrix} e^{i(kx+ly-\omega t)}$

<u>Polarization relations:</u>

(i) $\quad$ (5) $\Rightarrow \quad -i\omega\hat{b} + N^2\hat{w} = 0 \quad \Rightarrow \hat{b} = \frac{N^2}{i\omega}\hat{w} \qquad$ (or $\hat{w} = 0$ for $\omega = 0$).

(ii) $\quad$ (3) $\Rightarrow \quad \mu(-i\omega\hat{w}) = -\frac{1}{\rho_{oo}} \cdot im\hat{p} + \hat{b} \Rightarrow \hat{p} = -\frac{\rho_{oc}}{m\omega}\left\{N^2 - \mu\omega^2\right\}\hat{w}$.

(iii) $\quad$ (4) $\Rightarrow \quad i\vec{k}\cdot\hat{\vec{u}} = 0 \qquad \Rightarrow \hat{\vec{u}} \perp \vec{k}$ (motions $\perp$ to wavevector)

(iv) $\quad$ (1)+(2) $\Rightarrow$ (solving for u, v as in LSWE): $\quad \hat{u} = \frac{k\omega + ilf}{\omega^2 - f^2}\frac{\hat{p}}{\rho_{oo}}$

$$\hat{v} = \frac{l\omega - ikf}{\omega^2 - f^2}\frac{\hat{p}}{\rho_{oo}}$$

<u>Two possibilities</u>

(I) $\omega = 0$ (geostrophic mode): $\quad -fv_g = -\frac{1}{\rho_{oo}}\frac{\partial p'}{\partial x}, \quad fu_g = -\frac{1}{\rho_{oo}}\frac{\partial p'}{\partial y}, \quad \frac{\partial p'}{\partial z} = b, \quad \hat{w} = 0$

(II) Substitute (i), (ii), (iv) into (iii) to get one eqn for $\hat{w}$. Yields

$$\left[\frac{i(k^2+l^2)\omega}{\omega^2 - f^2}\cdot\left(-\frac{\rho_{oo}}{m\omega}\right)\left(N^2 - \mu\omega^2\right) + im\right]\hat{w} = 0$$

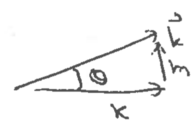
$$\Rightarrow \quad \omega^2 = \frac{(K^2N^2 + m^2f^2)}{(\mu K^2 + m^2)}, \quad K = (k^2+l^2)^{\frac{1}{2}} = \text{total hor. wavenumber}$$

(Dispersion reln. for Inertia-Gravity waves)

Hydrostatic approximation $\mu = 0$ valid if $L = \frac{2\pi}{K} \gg H = \frac{2\pi}{m}$, i.e. if $K \ll m$

In this case $\mu K^2 + m^2 \approx m^2$ so the neglected term really is small:

$$\underset{(K \ll m)}{\mu = 0}: \quad \omega^2 = f^2 + \frac{N^2K^2}{m^2}$$



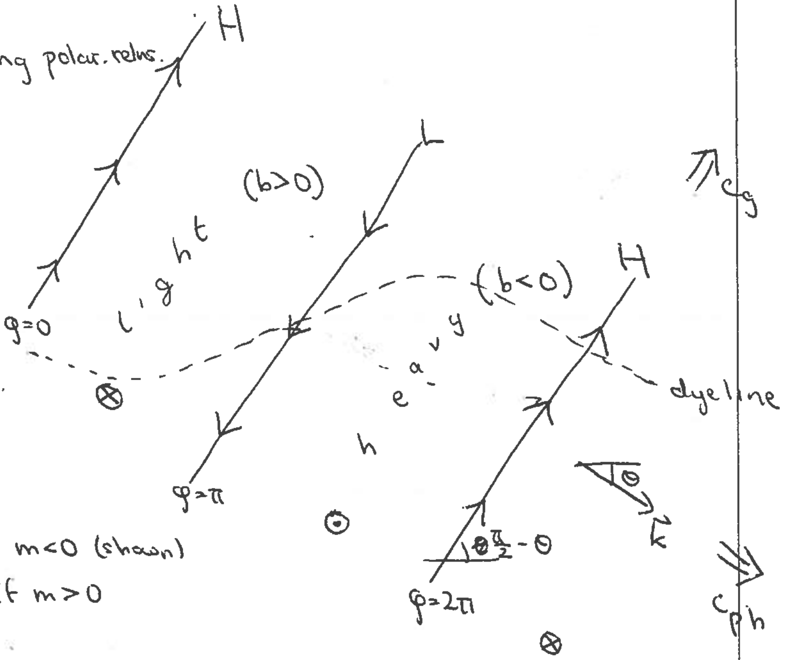
In general ($\mu = 1$), let $\theta = \measuredangle$ of $\vec{k}$ from horizontal, so

$\sin\theta = \frac{m}{(K^2+m^2)^{\frac{1}{2}}}$, $\cos\theta = \frac{K}{(K^2+m^2)^{\frac{1}{2}}}$. Then $\boxed{\omega^2 = N^2\cos^2\theta + f^2\sin^2\theta}$

depends only on <u>angle</u> of wavevector to horizontal. Note $f < |\omega| < N$.

Hydrostatic approx $\Rightarrow \theta \approx \frac{\pi}{2}$ ($\vec{k}$ near vertical, $\vec{u} \perp \vec{k}$ near horizontal).

Now, letting phase $\varphi = \vec{k}\cdot\vec{x} - \omega t$ + using polar. relns.
we get the following picture



- In particular, if $l=0$ and $k,\omega>0$ then:

$\hat{u} = \frac{k\omega}{\omega^2-f^2}\hat{p}$ in phase with $\hat{p}$

$\hat{v} = -\frac{ikf}{\omega^2-f^2}\hat{p}$ $\frac{\pi}{2}$ behind $\hat{p}$

$\hat{p} = -\frac{\rho_{00}}{m\omega}\left(N^2-\mu\omega^2\right)\hat{w}$ $\begin{cases}\text{in phase with } \hat{w} \text{ if } m<0 \text{ (shown)}\\ \text{antiphase with } \hat{w} \text{ if } m>0\end{cases}$

$\hat{b} = \frac{N^2}{i\omega}\hat{w}$ $\frac{\pi}{2}$ behind $\hat{w}$.

- Note displacement $z_p(x,y,z,t)$ has $\frac{Dz_p}{Dt}=w \Rightarrow$ (linearizing) $\frac{\partial z_p}{\partial t}=w$, $\hat{z}_p = \frac{\hat{w}}{-i\omega}$ is $\frac{\pi}{2}$ ahead of $\hat{w}$.
- Note $\frac{\hat{v}}{\hat{u}} = \frac{-if}{\omega}$, so parcels follow clockwise ellipses due to Coriolis turning added to the pressure forcing.
- Limits: $\theta=0$, $|\omega|=N$ (buoyancy oscillations, motions purely vertical)

$\theta=\frac{\pi}{2}$, $|\omega|=f$ (inertial oscillations, motions purely horizontal)

In both limits, pressure forces drop out.

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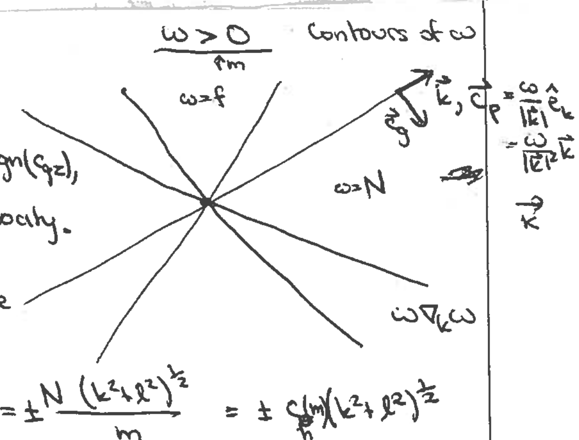
- Group velocity. $\vec{c}_g = \nabla_k\omega = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right)$



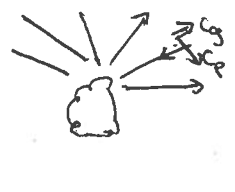
Looking at figure, we see contours of ω are along $\vec{k}$, so $\boxed{\vec{c}_g \perp \vec{k}}$. We also see that $\text{sign}(c_{ph\,z}) = -\text{sign}(c_{gz})$, upward group velocity requires downward phase velocity.
and larger $|\vec{k}| \Rightarrow$ small c_{ph}, c_g
Formulas in general are unredeeming, but in the hydrostatic, nonrotating limit ($f=\mu=0$)

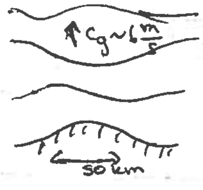
$$\omega^2 = \frac{N^2(k^2+l^2) + \cancel{f^2m^2}}{\cancel{\mu(k^2+l^2)} + m^2}, \quad \omega = \pm\frac{N(k^2+l^2)^{\frac{1}{2}}}{m} = \pm c_h(m)(k^2+l^2)^{\frac{1}{2}}$$

Thus in the horizontal, waves propagate nondispersively with phase speed = group velocity $c_h(m) = \frac{N}{m}$. For atm, $N=10^{-2}s^{-1}$, $m=\frac{\pi}{10^4 m} \Rightarrow c_h(m) = 30\frac{m}{s}$.
(For $f\neq0$, $|c_{gx}|<|c_{px}|$).
ocean, $N=5\times10^{-3}s^{-1}$, $m=\frac{\pi}{500m} \Rightarrow c_h(m) = 1\frac{m}{s}$.

☆ Note longer vertical wavelengths go faster.

In vertical, $c_g = \frac{\partial\omega}{\partial m} = \mp\underbrace{\frac{N}{m^2}(k^2+l^2)^{\frac{1}{2}}}_{k} \propto k$. For





atm, $k = \frac{\pi}{50\times10^3 m}$ for narrow mtn range $\Rightarrow c_{gz} = \frac{10^{-2}s^{-1}}{(\pi/10^4 m)^2}\cdot\frac{\pi}{50\times10^3 m}$
$= 6\frac{m}{s}$. $= 1600s$ for 10 km