

Continuously Stratified Flow (CR 9, Gill & A)

Assume a reference state  $\bar{p}(z), \bar{\rho}_\theta(z), \bar{\rho}(z)$ , etc. in hydrostatic balance. and let  $p' = p - \bar{p}$ , etc. The Boussinesq equations of motion on an  $f$  or  $\beta$  plane are:

(inviscid, adiabatic) 
$$\frac{D\vec{u}}{Dt} + f\hat{k} \times \vec{u} = -\frac{1}{\rho_{00}} \nabla p' + b\hat{k}, \quad b = -\frac{g\rho'_\theta}{\rho_{00}} = \text{buoyancy}$$

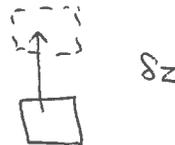
$$\nabla \cdot \vec{u} = 0$$

$$\frac{D\rho'_\theta}{Dt} = 0 \Rightarrow \frac{D\rho'_\theta}{Dt} + w \frac{d\bar{\rho}_\theta}{dz} = 0 \quad \text{or} \quad \frac{Db}{Dt} + \left(-\frac{g}{\rho_{00}} \frac{d\bar{\rho}_\theta}{dz}\right) w = 0$$

Note that the thermodynamics & dynamics interact thru buoyancy, which thus plays a pivotal role.

Stability

Consider displacing a fictional test parcel from  $z$  to  $z + \delta z$ . At this new height, what is its buoyancy?



$$b = -g \left\{ \frac{\rho_\theta(\text{parcel}) - \bar{\rho}_\theta(z + \delta z)}{\rho_{00}} \right\}, \quad \rho_\theta(\text{parcel}) = \bar{\rho}_\theta(z)$$

$$\Rightarrow \frac{g}{\rho_{00}} \frac{d\bar{\rho}_\theta}{dz} \delta z < 0 \quad \text{if} \quad \frac{d\bar{\rho}_\theta}{dz} < 0 \quad (\text{stably stratified})$$

If it were to move as a result of buoyancy alone,

$$\frac{D^2}{Dt^2} \delta z = \frac{Dw}{Dt} = b \Rightarrow$$

Thus, defining the buoyancy frequency  $N = \left(-\frac{g}{\rho_{00}} \frac{d\bar{\rho}_\theta}{dz}\right)^{\frac{1}{2}} \approx \left(-\frac{g}{\bar{\rho}_\theta} \frac{d\bar{\rho}_\theta}{dz}\right)^{\frac{1}{2}}$

we have  $\delta z = \delta z_0 \cos Nt$ ,  $\approx \left(\frac{g}{\bar{\theta}} \frac{d\bar{\theta}}{dz}\right)^{\frac{1}{2}} (\delta t m)$

parcel oscillates with frequency  $N$ .

Troposphere:  $N \sim \left(\frac{10 \frac{m}{s^2}}{300K} \cdot \frac{30K}{10^4 m}\right)^{\frac{1}{2}} = 10^{-2} s^{-1}$  ( $N \sim 2 \times 10^{-2} s^{-1}$  in stratosphere)

Thermocline:  $N \sim \left(\frac{10 \frac{m}{s^2}}{1028 \frac{kg}{m^3}} \cdot \frac{3 \frac{kg}{m^3}}{1000 m}\right)^{\frac{1}{2}} \sim 5 \times 10^{-3} s^{-1}$  ( $N < 10^{-3} s^{-1}$  in deep ocean)

Interbia-Gravity Waves on an f-plane (CR 10)

Assume small-amplitude disturbances on a stationary reference state. Then

$$\begin{aligned} (1) \quad u_t - fv &= -\frac{1}{\rho_{00}} p'_x \\ (2) \quad v_t + fu &= -\frac{1}{\rho_{00}} p'_y \\ (3) \quad \mu w_t &= -\frac{1}{\rho_{00}} p'_z + b \end{aligned}$$

$\mu = \begin{cases} 1 & \text{full eqns} \\ 0 & \text{hydrostatic approx} \end{cases}$

500 SHEETS FILLER 5 SQUARE  
50 SHEETS EYE-EASER 5 SQUARE  
100 SHEETS EYE-EASER 5 SQUARE  
70 SHEETS EYE-EASER 5 SQUARE  
100 RECYCLED WHITE 5 SQUARE  
42-399 200 RECYCLED WHITE 5 SQUARE  
42-399 200 RECYCLED WHITE 5 SQUARE  
13-782  
42-381  
42-382  
42-383  
42-384  
42-385  
42-386  
42-387  
42-388  
42-389  
42-390  
42-391  
42-392  
42-393  
42-394  
42-395  
42-396  
42-397  
42-398  
42-399  
42-400  
42-401  
42-402  
42-403  
42-404  
42-405  
42-406  
42-407  
42-408  
42-409  
42-410  
42-411  
42-412  
42-413  
42-414  
42-415  
42-416  
42-417  
42-418  
42-419  
42-420  
42-421  
42-422  
42-423  
42-424  
42-425  
42-426  
42-427  
42-428  
42-429  
42-430  
42-431  
42-432  
42-433  
42-434  
42-435  
42-436  
42-437  
42-438  
42-439  
42-440  
42-441  
42-442  
42-443  
42-444  
42-445  
42-446  
42-447  
42-448  
42-449  
42-450  
42-451  
42-452  
42-453  
42-454  
42-455  
42-456  
42-457  
42-458  
42-459  
42-460  
42-461  
42-462  
42-463  
42-464  
42-465  
42-466  
42-467  
42-468  
42-469  
42-470  
42-471  
42-472  
42-473  
42-474  
42-475  
42-476  
42-477  
42-478  
42-479  
42-480  
42-481  
42-482  
42-483  
42-484  
42-485  
42-486  
42-487  
42-488  
42-489  
42-490  
42-491  
42-492  
42-493  
42-494  
42-495  
42-496  
42-497  
42-498  
42-499  
42-500  
42-501  
42-502  
42-503  
42-504  
42-505  
42-506  
42-507  
42-508  
42-509  
42-510  
42-511  
42-512  
42-513  
42-514  
42-515  
42-516  
42-517  
42-518  
42-519  
42-520  
42-521  
42-522  
42-523  
42-524  
42-525  
42-526  
42-527  
42-528  
42-529  
42-530  
42-531  
42-532  
42-533  
42-534  
42-535  
42-536  
42-537  
42-538  
42-539  
42-540  
42-541  
42-542  
42-543  
42-544  
42-545  
42-546  
42-547  
42-548  
42-549  
42-550  
42-551  
42-552  
42-553  
42-554  
42-555  
42-556  
42-557  
42-558  
42-559  
42-560  
42-561  
42-562  
42-563  
42-564  
42-565  
42-566  
42-567  
42-568  
42-569  
42-570  
42-571  
42-572  
42-573  
42-574  
42-575  
42-576  
42-577  
42-578  
42-579  
42-580  
42-581  
42-582  
42-583  
42-584  
42-585  
42-586  
42-587  
42-588  
42-589  
42-590  
42-591  
42-592  
42-593  
42-594  
42-595  
42-596  
42-597  
42-598  
42-599  
42-600  
42-601  
42-602  
42-603  
42-604  
42-605  
42-606  
42-607  
42-608  
42-609  
42-610  
42-611  
42-612  
42-613  
42-614  
42-615  
42-616  
42-617  
42-618  
42-619  
42-620  
42-621  
42-622  
42-623  
42-624  
42-625  
42-626  
42-627  
42-628  
42-629  
42-630  
42-631  
42-632  
42-633  
42-634  
42-635  
42-636  
42-637  
42-638  
42-639  
42-640  
42-641  
42-642  
42-643  
42-644  
42-645  
42-646  
42-647  
42-648  
42-649  
42-650  
42-651  
42-652  
42-653  
42-654  
42-655  
42-656  
42-657  
42-658  
42-659  
42-660  
42-661  
42-662  
42-663  
42-664  
42-665  
42-666  
42-667  
42-668  
42-669  
42-670  
42-671  
42-672  
42-673  
42-674  
42-675  
42-676  
42-677  
42-678  
42-679  
42-680  
42-681  
42-682  
42-683  
42-684  
42-685  
42-686  
42-687  
42-688  
42-689  
42-690  
42-691  
42-692  
42-693  
42-694  
42-695  
42-696  
42-697  
42-698  
42-699  
42-700  
42-701  
42-702  
42-703  
42-704  
42-705  
42-706  
42-707  
42-708  
42-709  
42-710  
42-711  
42-712  
42-713  
42-714  
42-715  
42-716  
42-717  
42-718  
42-719  
42-720  
42-721  
42-722  
42-723  
42-724  
42-725  
42-726  
42-727  
42-728  
42-729  
42-730  
42-731  
42-732  
42-733  
42-734  
42-735  
42-736  
42-737  
42-738  
42-739  
42-740  
42-741  
42-742  
42-743  
42-744  
42-745  
42-746  
42-747  
42-748  
42-749  
42-750  
42-751  
42-752  
42-753  
42-754  
42-755  
42-756  
42-757  
42-758  
42-759  
42-760  
42-761  
42-762  
42-763  
42-764  
42-765  
42-766  
42-767  
42-768  
42-769  
42-770  
42-771  
42-772  
42-773  
42-774  
42-775  
42-776  
42-777  
42-778  
42-779  
42-780  
42-781  
42-782  
42-783  
42-784  
42-785  
42-786  
42-787  
42-788  
42-789  
42-790  
42-791  
42-792  
42-793  
42-794  
42-795  
42-796  
42-797  
42-798  
42-799  
42-800  
42-801  
42-802  
42-803  
42-804  
42-805  
42-806  
42-807  
42-808  
42-809  
42-810  
42-811  
42-812  
42-813  
42-814  
42-815  
42-816  
42-817  
42-818  
42-819  
42-820  
42-821  
42-822  
42-823  
42-824  
42-825  
42-826  
42-827  
42-828  
42-829  
42-830  
42-831  
42-832  
42-833  
42-834  
42-835  
42-836  
42-837  
42-838  
42-839  
42-840  
42-841  
42-842  
42-843  
42-844  
42-845  
42-846  
42-847  
42-848  
42-849  
42-850  
42-851  
42-852  
42-853  
42-854  
42-855  
42-856  
42-857  
42-858  
42-859  
42-860  
42-861  
42-862  
42-863  
42-864  
42-865  
42-866  
42-867  
42-868  
42-869  
42-870  
42-871  
42-872  
42-873  
42-874  
42-875  
42-876  
42-877  
42-878  
42-879  
42-880  
42-881  
42-882  
42-883  
42-884  
42-885  
42-886  
42-887  
42-888  
42-889  
42-890  
42-891  
42-892  
42-893  
42-894  
42-895  
42-896  
42-897  
42-898  
42-899  
42-900  
42-901  
42-902  
42-903  
42-904  
42-905  
42-906  
42-907  
42-908  
42-909  
42-910  
42-911  
42-912  
42-913  
42-914  
42-915  
42-916  
42-917  
42-918  
42-919  
42-920  
42-921  
42-922  
42-923  
42-924  
42-925  
42-926  
42-927  
42-928  
42-929  
42-930  
42-931  
42-932  
42-933  
42-934  
42-935  
42-936  
42-937  
42-938  
42-939  
42-940  
42-941  
42-942  
42-943  
42-944  
42-945  
42-946  
42-947  
42-948  
42-949  
42-950  
42-951  
42-952  
42-953  
42-954  
42-955  
42-956  
42-957  
42-958  
42-959  
42-960  
42-961  
42-962  
42-963  
42-964  
42-965  
42-966  
42-967  
42-968  
42-969  
42-970  
42-971  
42-972  
42-973  
42-974  
42-975  
42-976  
42-977  
42-978  
42-979  
42-980  
42-981  
42-982  
42-983  
42-984  
42-985  
42-986  
42-987  
42-988  
42-989  
42-990  
42-991  
42-992  
42-993  
42-994  
42-995  
42-996  
42-997  
42-998  
42-999  
42-1000



Made in U.S.A.

$$(4) \quad u_x + v_y + w_z = 0$$

$$(5) \quad \frac{\partial b}{\partial t} + N^2 w = 0$$

If  $N^2$  is constant in  $z$ , expect solutions 
$$\begin{bmatrix} u \\ v \\ w \\ b \\ p \end{bmatrix} = \text{Re} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{b} \\ \hat{p} \end{bmatrix} e^{i(kx + ly - \omega t)}$$

Polarization relations:

$$(i) \quad (5) \Rightarrow -i\omega \hat{b} + N^2 \hat{w} = 0 \Rightarrow \hat{b} = \frac{N^2}{i\omega} \hat{w} \quad (\text{or } \hat{w} = 0 \text{ for } \omega = 0)$$

$$(ii) \quad (3) \Rightarrow \mu(-i\omega \hat{w}) = -\frac{1}{\rho_0} \cdot i m \hat{p} + \hat{b} \Rightarrow \hat{p} = -\frac{\rho_0}{m\omega} \{N^2 - \mu\omega^2\} \hat{w}$$

$$(iii) \quad (4) \Rightarrow i\vec{k} \cdot \vec{\hat{u}} = 0 \Rightarrow \vec{\hat{u}} \perp \vec{k} \quad (\text{motions } \perp \text{ to wavevector})$$

$$(iv) \quad (1)+(2) \Rightarrow (\text{solving for } u, v \text{ as in LSWE}): \quad \hat{u} = \frac{k\omega + i l f}{\omega^2 - f^2} \frac{\hat{p}}{\rho_0}$$

$$\hat{v} = \frac{l\omega - i k f}{\omega^2 - f^2} \frac{\hat{p}}{\rho_0}$$

Two possibilities

$$(I) \quad \omega = 0 \quad (\text{geostrophic mode}): \quad -f v_g = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}, \quad f u_g = -\frac{1}{\rho_0} \frac{\partial p'}{\partial y}, \quad \frac{\partial p'}{\partial z} = b, \quad w^3 = 0$$

(II) Substitute (i), (ii), (iv) into (iii) to get one eqn for  $\hat{w}$ . Yields

$$\left[ \frac{i(k^2 + l^2)\omega}{\omega^2 - f^2} \cdot \left(-\frac{\rho_0}{m\omega}\right) (N^2 - \mu\omega^2) + i m \right] \hat{w} = 0$$

$$\Rightarrow \omega^2 = \frac{(k^2 N^2 + m^2 f^2)}{(\mu k^2 + m^2)} \quad , \quad k = (k^2 + l^2)^{\frac{1}{2}} = \text{total hor. wavenumber} \\ (\text{Dispersion reln. for Inertia-Gravity waves})$$

Hydrostatic approximation  $\mu = 0$  valid if  $L = \frac{2\pi}{k} \gg H = \frac{2\pi}{m}$ , i.e. if  $k \ll m$

In this case  $\mu k^2 + m^2 \approx m^2$  so the neglected term really is small:

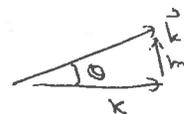
$$\mu = 0: \quad \omega^2 = f^2 + \frac{N^2 k^2}{m^2} \quad (k \ll m)$$

In general ( $\mu = 1$ ), let  $\theta = \angle$  of  $\vec{k}$  from horizontal, so

$$\sin \theta = \frac{m}{(k^2 + m^2)^{\frac{1}{2}}}, \quad \cos \theta = \frac{k}{(k^2 + m^2)^{\frac{1}{2}}}. \quad \text{Then } \boxed{\omega^2 = N^2 \cos^2 \theta + f^2 \sin^2 \theta}$$

depends only on angle of wavevector to horizontal. Note  $f < |\omega| < N$ .

Hydrostatic approx  $\Rightarrow \theta \approx \frac{\pi}{2}$  ( $\vec{k}$  near vertical,  $\vec{u} \perp \vec{k}$  near horizontal).



Now, letting phase  $\phi = \vec{k} \cdot \vec{x} - \omega t$  using polar coords. we get the following picture

In particular, if  $l=0$  and  $k, \omega > 0$  then:

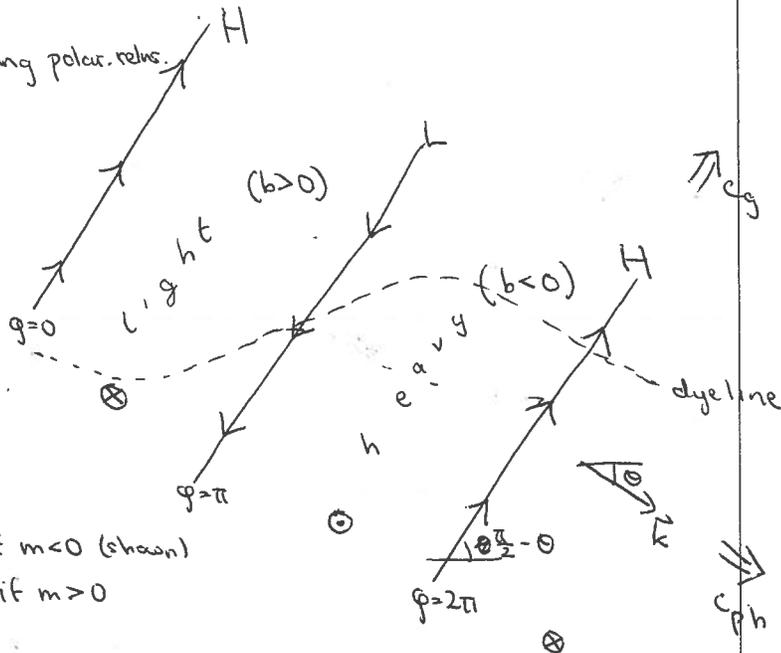
$$\hat{u} = \frac{k\omega}{\omega^2 - f^2} \hat{p} \quad \text{in phase with } \hat{p}$$

$$\hat{v} = -\frac{ikf}{\omega^2 - f^2} \hat{p} \quad \frac{\pi}{2} \text{ behind } \hat{p}$$

$$\hat{p} = -\frac{\rho_0 \omega}{m\omega(N^2 - \mu\omega^2)} \hat{w}$$

{ in phase with  $\hat{w}$  if  $m < 0$  (shown)  
 { antiphase with  $\hat{w}$  if  $m > 0$

$$\hat{b} = \frac{N^2}{\omega} \hat{w} \quad \frac{\pi}{2} \text{ behind } \hat{w}$$



Note displacement  $z_p(x,y,z,t)$  has  $\frac{\partial z_p}{\partial t} = \omega \Rightarrow$  (linearizing)  $\frac{\partial z_p}{\partial t} = \omega$ ,  $\hat{z}_p = \frac{\hat{w}}{-i\omega}$  is  $\frac{\pi}{2}$  ahead of  $\hat{w}$ .

Note  $\frac{\hat{v}}{\hat{u}} = -\frac{if}{\omega}$ , so parcels follow clockwise ellipses due to Coriolis turning added to the pressure forcing.

- Limits:  $\theta=0, |\omega|=N$  (buoyancy oscillations, motions purely vertical)
- $\theta=\frac{\pi}{2}, |\omega|=f$  (inertial oscillations, motions purely horizontal)

In both limits, pressure forces drop out.

Group velocity.  $\hat{c}_g = \nabla_k \omega = \left( \frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right)$

Looking at figure, we see contours of  $\omega$  are along  $\vec{k}$ , so  $\hat{c}_g \perp \vec{k}$ . We also see that  $\text{sign}(c_{phz}) = -\text{sign}(c_{gz})$ .

upward group velocity requires downward phase velocity.

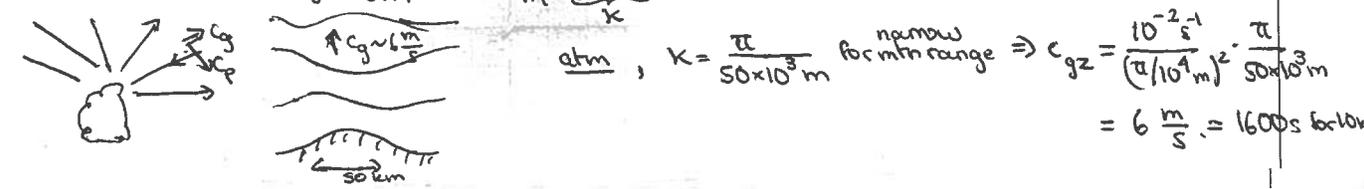
and larger  $|\vec{k}| \Rightarrow$  small  $c_{ph}, c_g$   
 Formulas in general are unredeeming, but in the hydrostatic, nonrotating limit ( $f=\mu=0$ )

$$\omega^2 = \frac{N^2(k^2 + l^2)}{\mu(k^2 + l^2)} + \frac{f^2 m^2}{m^2}, \quad \omega = \pm \frac{N(k^2 + l^2)^{1/2}}{m} = \pm \frac{c_p}{m} (k^2 + l^2)^{1/2}$$

Thus in the horizontal, waves propagate nondispersively with phase speed = group velocity  $c_h(m) = \frac{N}{m}$ . For atm,  $N=10^{-2} s^{-1}$ ,  $m = \frac{\pi}{10^4 m} \Rightarrow c_h(m) = 30 \frac{m}{s}$ .  
 (For  $f \neq 0, |c_{gx}| < |c_{px}|$ ). ocean,  $N=5 \times 10^{-3} s^{-1}$ ,  $m = \frac{\pi}{500 m} \Rightarrow c_h(m) = 1 \frac{m}{s}$ .

\* Note longer vertical wavelengths go faster.

In vertical,  $c_g = \frac{\partial \omega}{\partial m} = \mp \frac{N}{m^2} (k^2 + l^2)^{1/2} \propto \kappa$ . For  $\kappa$



atm,  $\kappa = \frac{\pi}{50 \times 10^3 m}$  narrow form m range  $\Rightarrow c_{gz} = \frac{10^{-2} s^{-1}}{(\pi/10^4 m)^2} \cdot \frac{\pi}{50 \times 10^3 m} = 6 \frac{m}{s} = 1600 s \text{ below}$

13-782 500 SHEETS, FILLER, 5 SQUARE  
 42-381 50 SHEETS, EYE GLASS, 5 SQUARE  
 42-382 100 SHEETS, EYE GLASS, 5 SQUARE  
 42-383 100 SHEETS, EYE GLASS, 5 SQUARE  
 42-392 100 RECYCLED WHITE, 5 SQUARE  
 42-399 200 RECYCLED WHITE, 5 SQUARE  
 Made in U.S.A.



120