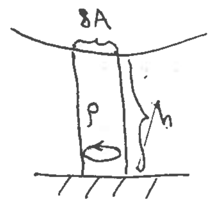
Potential Vorticity

Potential Vorticity — Circulation thm / Mass conservation

Homogeneous shallow water layer



Circulation thm $\Rightarrow \Gamma_a = (\zeta + f)\,\delta A$ conserved

Mass cons. $\Rightarrow \delta m = \rho h\,\delta A$ conserved

$$\Rightarrow \frac{\Gamma_a}{\delta m} = \frac{1}{\rho}\left(\frac{\zeta+f}{h}\right) \text{ conserved}$$

$$\Rightarrow q = \frac{\zeta + f}{h} \quad \text{conserved since } \rho \text{ constant.}$$

Continuously stratified fluid

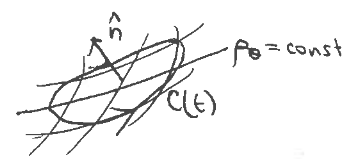
$$= \int_{C(t)} [\vec{u} + \vec{\Omega}\times\vec{x}]\cdot d\vec{x}$$

$$\nabla\cdot(\vec{\Omega}\times\vec{x}) = \vec{\Omega}\,\nabla\cdot\vec{x} - \vec{x}\,\nabla\cdot\vec{\Omega}$$

Circulation Thm: If $\Gamma_a = \int_{C(t)} \vec{u}\cdot d\vec{x}$ ~~and~~ $+ \int_{int(C)} 2\vec{\Omega}\cdot\hat{n}\, dA$, then

$$\frac{D\Gamma_a}{Dt} = \int_{int(C)} \frac{\nabla\rho\times\nabla p}{\rho^2}\cdot\hat{n}\, dA$$

Special case: If $C(t)$ lies entirely in the same potential density surface, then Γ_a is conserved, assuming that
$\rho_\theta = F(\rho, p)$ (OK for dry air, fresh water. $\approx$ OK for moist air, salty water).



since then $\hat{n} = \dfrac{\nabla\rho_\theta}{|\nabla\rho_\theta|}$ is parallel to $\nabla\rho_\theta$, and
thus

$$\hat{n} = \frac{1}{|\nabla\rho_\theta|}\left\{\frac{\partial F}{\partial\rho}\nabla\rho + \frac{\partial F}{\partial p}\nabla p\right\}$$

so

$$(\nabla\rho\times\nabla p)\cdot\hat{n} = 0.$$

Furthermore if C starts in the surface $\rho_\theta = $ const, it remains so due to potl density conservation.

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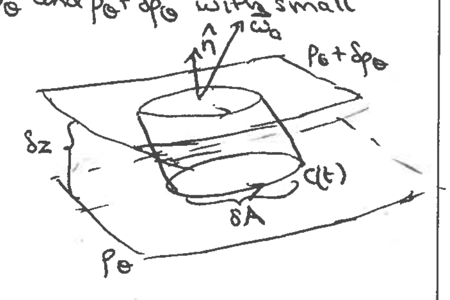
Application: Consider a thin slab of fluid sandwiched between ρ_θ and $\rho_\theta + \delta\rho_\theta$ with small cross sectional area $\delta A(t)$ letting $C(t)$ be a circuit around its bottom face,



$$\Gamma_a = \int_{int(C)} \vec{\omega}_a\cdot\hat{n}\, dA = \vec{\omega}_a\cdot\hat{n}\,\delta A(t) = \text{const.}$$

Furthermore, its mass

$$\delta m = \rho(t)\,\delta A(t)\,\delta z(t) = \text{const.}$$

Lastly

$$\hat{n} = \frac{\nabla\rho_\theta}{|\nabla\rho_\theta|} \quad \text{and} \quad \delta z = \frac{\delta\rho_\theta}{|\nabla\rho_\theta|}$$

Thus

$$\Gamma_a = \vec{\omega}_a\cdot\frac{\nabla\rho_\theta}{|\nabla\rho_\theta|}\cdot\frac{\delta m}{\rho\,\delta z} = \frac{\vec{\omega}_a\cdot\nabla\rho_\theta}{\rho}\cdot\frac{\delta m}{\delta\rho_\theta}$$

$\Rightarrow$ Ertel PV $\quad q \equiv \dfrac{\vec{\omega}_a\cdot\nabla\rho_\theta}{\rho}$ is const following fluid. $\left(\dfrac{Dq}{Dt} = 0\right)$.

$=$ vorticity $\cdot$ stratification. (column ~~[illegible]~~ height changes reflected in stratification).

No motion (or low Ro) $\Rightarrow$ $q_0 = -\frac{f}{\rho}\frac{\partial \rho_\theta}{\partial z} \approx \frac{fN^2}{g} \propto$ Coriolis param (Ambient vorticity) × Static Stability.

Atmosphere: $q_\theta = \frac{\vec{\omega}_a \cdot \nabla\theta}{\rho}$ const.

Boussinesq: $q = -\frac{\vec{\omega}_a \cdot \nabla\rho_\theta}{\rho_0}$ const.

Linearized $\zeta \ll f$, $\frac{db}{dz} \ll N^2$

$q = q_0 + q'$; $q' = q_0\left\{1 + \frac{\zeta}{f} + \frac{1}{N^2}\frac{db}{dz}\right\}$.

Hydrostatic approx: Replace $\vec{u} = (u,v,w)$ by $\vec{u}_h = (u,v,0)$ and compute $\vec{\omega} = \nabla \times \vec{u}_h$.

PV as a tracer

On timescales (periods of a few days in atm, a few weeks in ocean) on which diabatic, viscous effects are secondary, q is a flow tracer.

Stirred regions $\longleftrightarrow$ Fairly homogenized q

Streamline of steady flow $\longleftrightarrow$ constant q

Dynamically separate regions $\longleftrightarrow$ large differences in q.

Of course q is a special flow tracer, since its distribution constrains (and for low Ro flow, determines) the flow itself. Note that for low Ro flow, $q \approx -\frac{f}{\rho}\frac{\partial\rho_\theta}{\partial z}$ is related to static stability & coriolis parameter.

Examples: Troposphere (low PV) – stratosphere (high PV)

Polar vortex in stratosphere – "surf zone"

N Pacific vertical + horizontal ρ_θ, PV sections.

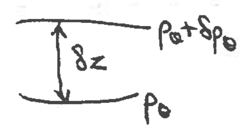
Viewgraphs

skip

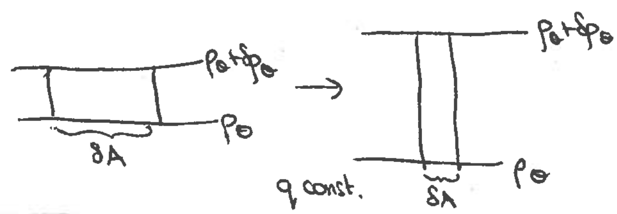
use as HW.

Interpretation: As columns stretch or squash, $\zeta + f$ increases or decreases in proportion. Column stretch is manifest in decreased stratification as potl. density surfaces are pulled apart.

$\delta z = \frac{\delta\rho_\theta}{\left(-\frac{d\rho_\theta}{dz}\right)}$



Low $\zeta + f$
Low δz
High $\frac{d\rho_\theta}{dz}$
High δA



High $\zeta + f$
High δz
Low $\frac{d\rho_\theta}{dz}$
Low δA.

GFD

PV Climatology - Atmosphere

James, Intro. to Circulating Atmospheres

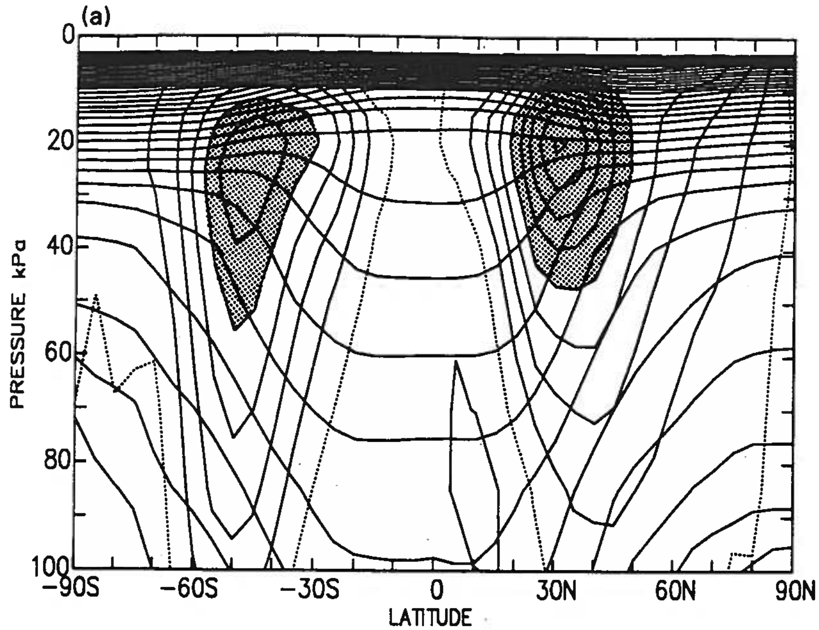


Fig. 4.2. Contours of $[\overline{u}]$ and $[\overline{\theta}]$ for (a) DJF; and (b) JJA, based on six years of ECMWF data. Contour interval for $[\overline{u}]$ as in Fig. 4.1. Contour interval for θ is 5K.
(5 m/s)

$$q \approx \frac{1}{\rho}(\zeta+f)\frac{\partial\theta}{\partial z}$$

$$\approx \frac{f}{\rho}\frac{\partial\theta}{\partial z}$$ (on large lengthscales)

since $\frac{\zeta}{f} = O(Ro) \sim 0.1$ in midlatitudes

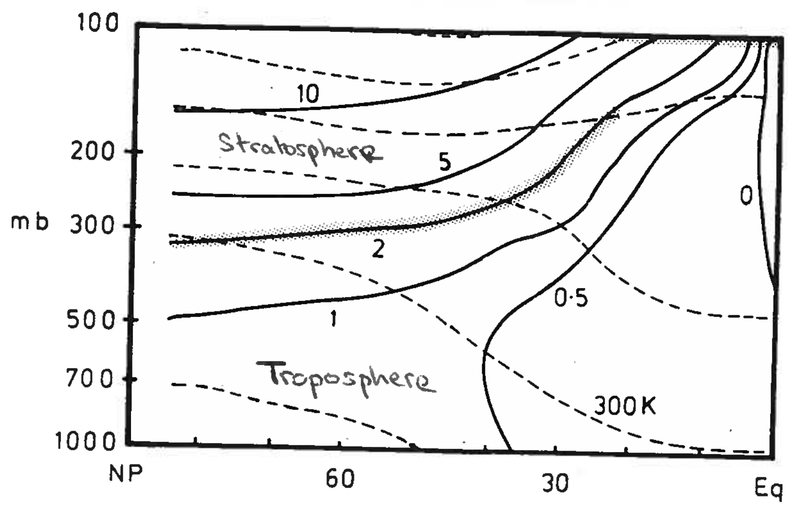


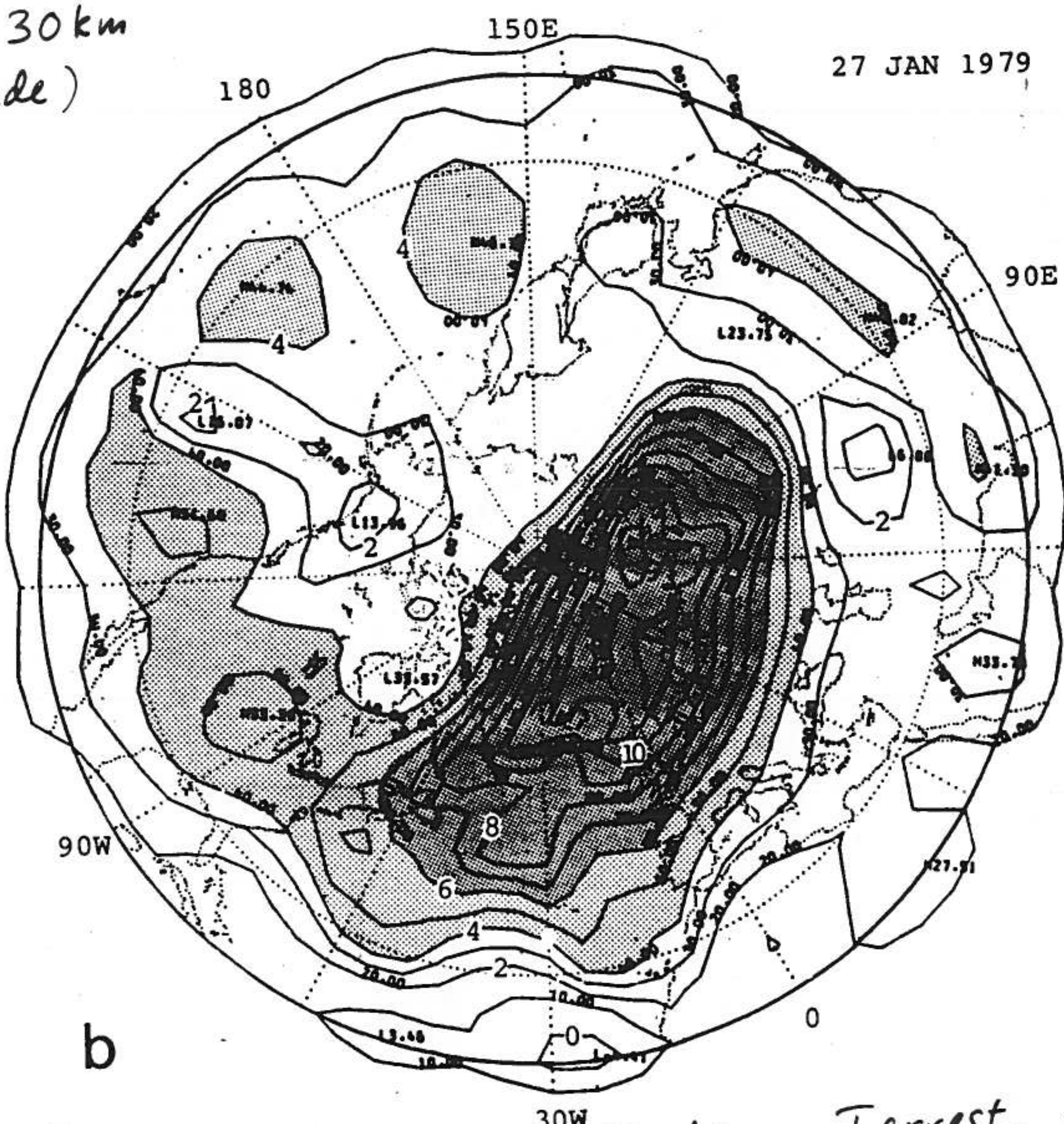
FIG. 5.5. A summary of the climatological PV and θ distribution below 100 mb in the Northern Hemisphere winter. The dashed contours are those of θ, drawn every 30 K. Values of PV are given in terms of the unit PVU = 10^{-6} K m^2 kg^{-1} s^{-1}, and contours are drawn at 0, 0.5, 1, 2, 5 and 10. The "dynamical" tropopause, specified by 2 PVU, is indicated by stippling.

Stratospheric Polar Vortex
("Leaking" into a "surf zone")

Isentropic map of Rossby-Ertel potential vorticity on 850°K isentropic surface

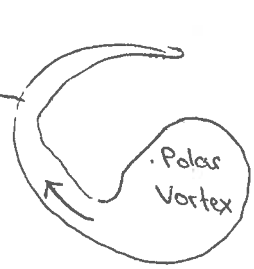
$\rho^{-1}(2\underset{\sim}{\Omega}+\nabla\times\underset{\sim}{u})\cdot\nabla\theta$

(near 30km altitude)



(McIntyre & Palmer 1984 J. Atmos. Terrest. Phys. 46, 825)

"Surf Zone" of leaked high PV air.



$\left(q \approx -\frac{f}{\rho}\frac{\partial \rho}{\partial z}\right)$

PV Climatology - Oceans

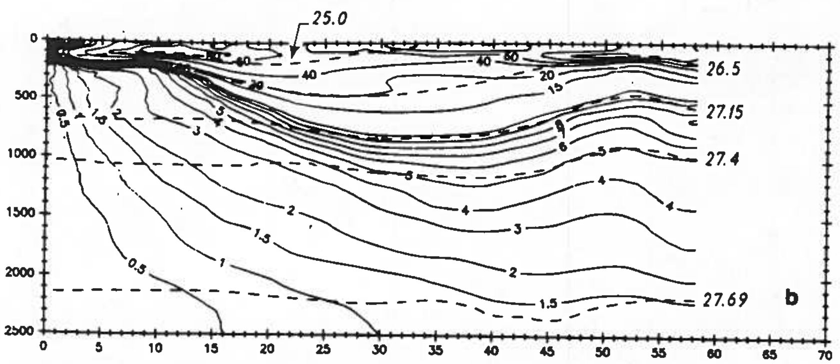


Figure 17 Meridional sections of North Pacific potential vorticity at (*a*) 150°E longitude and (*b*) 150°W longitude. The intensity of the circulation decreases to the east, but the parallelism of Q (solid) and ρ (dashed) remains. The gyre boundary seems to lean poleward with increasing depth.

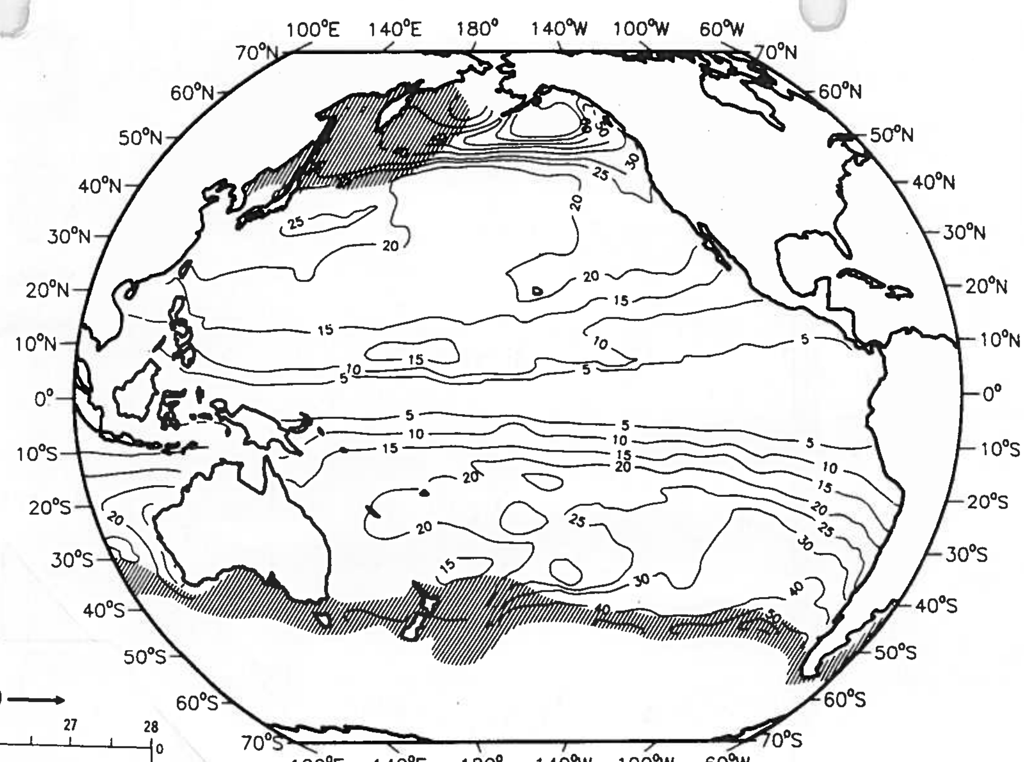


Figure 16b Potential vorticity for $\rho = 1.0265$ (typically 500 m depth). The figure is dominated by nearly uniform potential vorticity from Asia to North America, as in the numerical simulations and the recirculation theory. In addition, one sees inward advection of tongues of large q in the east and small "islands" (possibly due to buoyancy forcing) in the west. A buoyancy-driven island dominates the subpolar gyre. The frontal variation in q between subpolar and subtropical gyres agrees with the uppermost layer of the simulation (Figure 13e).

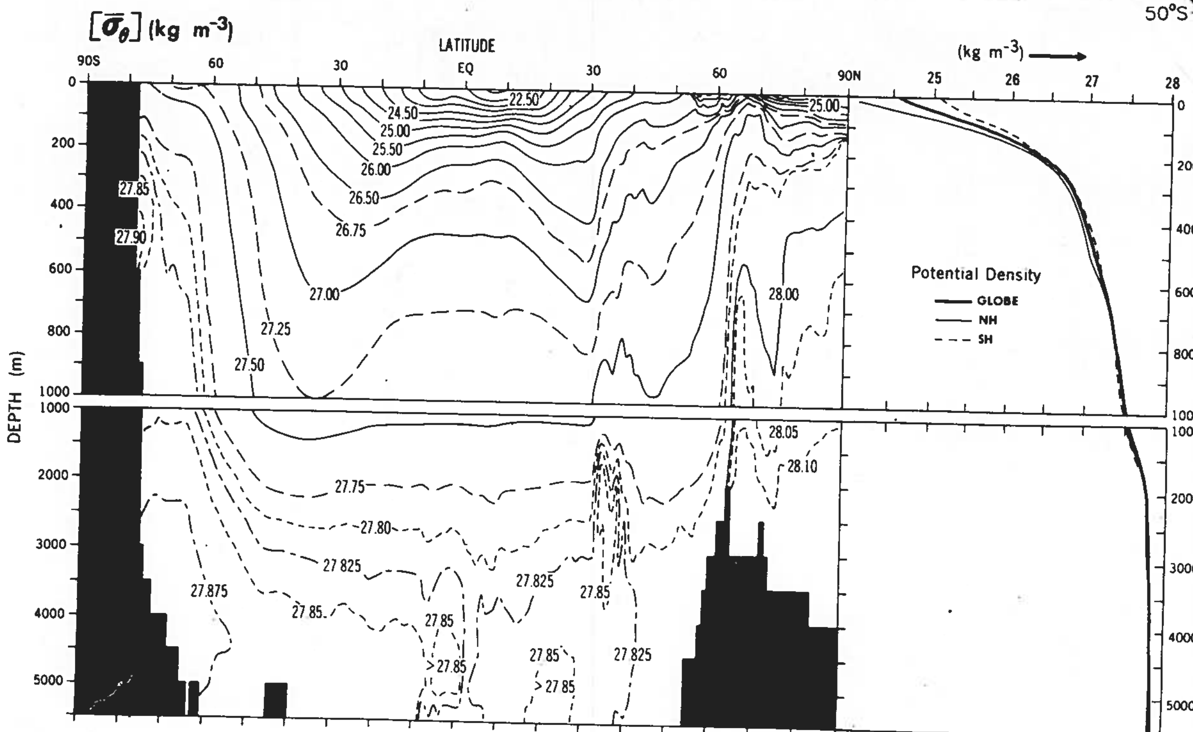


FIGURE 8.14. Zonal-mean cross section of sigma-t for the top ocean layer 0–1000 m and the layer below 1000-m depth in kg m^{-3} for annual-mean conditions (after Levitus, 1982). Vertical profiles of the hemispheric and global mean sigma-t are shown on the right.

Peixoto and Oort, Physics of Climate, p. 195