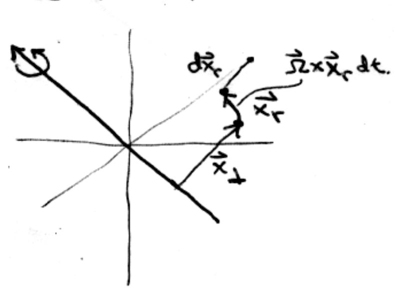
Rotating reference frame (Gill 4.5.2) 12

Let $\vec{x}_f(t)$ be the position of a mass point in a fixed 'inertial' reference frame

Let $\vec{x}_r(t)$ be its position in a reference frame rotating with ~~a~~ velocity vector $\vec{\Omega}$ which is coincident with the fixed frame at $t=0$.

At time dt a point $\vec{x}_r$ in the rotating ref frame will have spun $\Omega \times \vec{x}_r\,dt$ in the fixed frame, so



$$d\vec{x}_f = \vec{\Omega}\times\vec{x}_r\,dt + d\vec{x}_r$$

so

$$\vec{u}_f \equiv \frac{d\vec{x}_f}{dt} = \frac{d\vec{x}_r}{dt} + \Omega\times\vec{x}_r$$

Repeating this argument on $\vec{u}_f(t)$

$$\frac{d\vec{u}_f}{dt} = \frac{d}{dt}\left\{\frac{d\vec{x}_r}{dt} + \Omega\times\vec{x}_r\right\} + \vec{\Omega}\times\left\{\frac{d\vec{x}_r}{dt} + \vec{\Omega}\times\vec{x}_r\right\}$$

$$= \frac{d^2\vec{x}_r}{dt} + 2\vec{\Omega}\times\frac{d\vec{x}_r}{dt} + \vec{\Omega}\times(\vec{\Omega}\times\underbrace{\vec{x}_r}_{\vec{x}_\perp+\vec{x}_\parallel})$$

Define $\vec{u}_r = \dfrac{d\vec{x}_r}{dt}$ as the velocity relative to the rotating frame. Then

$$\frac{d\vec{u}_f}{dt} = \frac{d\vec{u}_r}{dt} + 2\vec{\Omega}\times\vec{u}_r - \Omega^2\vec{x}_\perp$$

(since $\vec{\Omega}\times\vec{x}_\parallel = 0$ and $\vec{\Omega}, \vec{x}_\perp$ and $\vec{\Omega}\times\vec{x}_\perp$ are mutually orthogonal.)

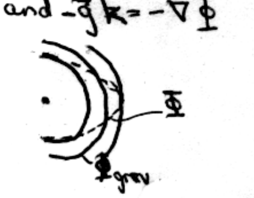
Regarding a fluid parcel as a mass point and using Newton's second law,

$$\frac{d\vec{u}_f}{dt} = \text{force/mass}$$

$$\frac{d\vec{u}_r}{dt} + 2\vec{\Omega}\times\vec{u}_r = -\frac{1}{\rho}\nabla p - \nabla\phi_{grav} + \Omega^2\vec{x}_\perp + \vec{F}(\vec{u}_f).$$

Note that $\Omega^2\vec{x}_\perp = -\nabla(\underbrace{\tfrac{1}{2}\Omega^2 r_\perp^2}_{\phi_{centrifugal}})$ where $r_\perp$ is distance from rotation axis

so letting $\Phi = \underbrace{\phi_{grav}}_{-\frac{GM_e}{R}} + \phi_{centrifugal}$ be the "geopotential" and $-g\hat{k} = -\nabla\Phi$



we get (dropping 'r')

$$\frac{d\vec{u}}{dt} + 2\vec{\Omega}\times\vec{u} = -\frac{1}{\rho}\nabla p - g\hat{k} + \vec{F}(\vec{u})$$

... identical to fixed frame except for apparent Coriolis accel. $-2\vec{\Omega}\times\vec{u}$

Mean sea level $z=0$ is defined as a geopotential surface, so $\Phi = gz$.

At equator $|-\nabla\phi_{centrifugal}| = \Omega^2 r_0 = \left(\dfrac{2\pi}{86400\,s}\right)^2(6400\times10^3\,m) = 0.03\,\frac{m}{s^2}$ (0.3% of g)

and the geoid is ~ 20 km (0.3%) further from center of earth.

Coriolis accel has magnitude $\Omega U \sim 10^{-4}\,s^{-1}\cdot\begin{cases}10 & \text{atm}\\ 1 & \text{ocn}\end{cases} \le 10^{-3}\,\frac{m}{s^2}$ $\left(100\,\frac{m}{s\text{-}day}\right)$

which is smaller yet, but adds up over periods of hours.

Scaling

$$\frac{\left[\frac{Du}{Dt}\right]}{[2\vec{\Omega}\times\vec{u}]} = \frac{\frac{U^2}{L}}{fU} = \frac{U}{fL} \equiv Ro \text{, the Rossby number.}$$

For $U = \begin{pmatrix}10 \frac{m}{s}\\ 1 \frac{m}{s}\end{pmatrix}$, $L = \begin{pmatrix}10^6 m\\ 10^5 m\end{pmatrix}$, $Ro = 0.1 \Rightarrow$ Coriolis accelerations dominate inertial accelerations.

Interpretation

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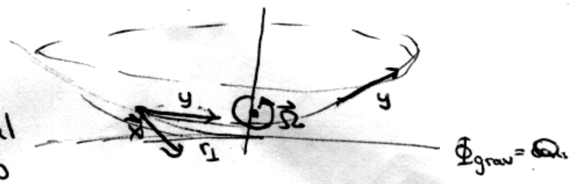
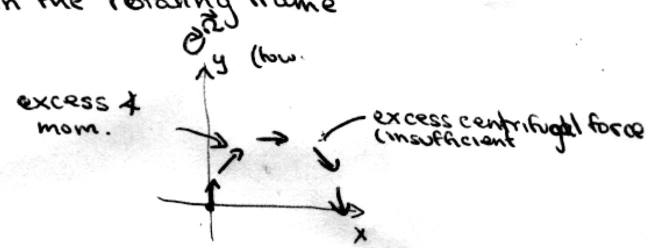
Think of fluid on a geopotl surface or geoid as spinning around a paraboloidal dish, curved to allow



equilibrium circular orbits at angular velocity Ω and ∡ mom. $\Omega r_\perp^2$.

A fluid parcel with initial ∡ velocity $\vec{\Omega}$ but moving inward will conserve ∡ mom. and start to spin faster, bending to right in the rotating frame



A fluid parcel with ∡ velocity faster than $\vec{\Omega}$ will push outward as the sloped dish provides insufficient centripetal acceleration. The result is that moving parcel always experience a rightward accel. $-2\vec{\Omega}\times\vec{u}$ if $\vec{\Omega} = \Omega\hat{k}$ and leftward if direction is opposite.

L8

Free motion of a mass point on a geoid ($\nabla p = F(\vec{u}) = 0$)

$$\frac{d\vec{x}}{dt} = \vec{u} \qquad \vec{u} = (u, v, 0).$$

$$\frac{d\vec{u}}{dt} = -2\vec{\Omega}\times\vec{u} \qquad = -2\Omega\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & \cos\phi & \sin\phi\\ u & v & 0\end{bmatrix}$$

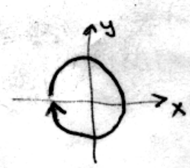
$$\Rightarrow \frac{du}{dt} = -\underbrace{2\Omega\sin\phi}_{f}\cdot v$$

$$\frac{dv}{dt} = \underbrace{2\Omega\sin\phi}_{f}\cdot u \qquad , f = 2\Omega\sin\phi = \text{Coriolis parameter}, \sim 10^{-4} s^{-1}.$$

[vertical force is balanced by hor. uniform pressure gradient]

$$\frac{d^2u}{dt^2} = -f^2 u \quad , u = v_0 \sin ft \quad , x = -\frac{v_0}{f}\cos ft$$

$$v = v_0\cos ft \quad , y = \frac{v_0}{f}\sin ft$$



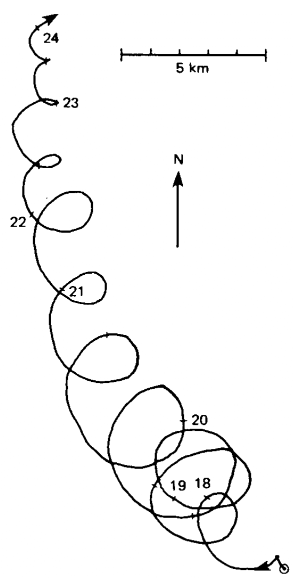
("Inertial oscillation")

For $v_0 = \begin{pmatrix}1\\ 10\end{pmatrix}\frac{m}{s}$, radius $= \frac{v_0}{f} = \begin{pmatrix}10^4\\ 10^5\end{pmatrix} m = \begin{pmatrix}10\\ 100\end{pmatrix} km$.

Examples of inertial oscillations

Ocean – forced by rapid changes in surface wind stress as weather systems pass

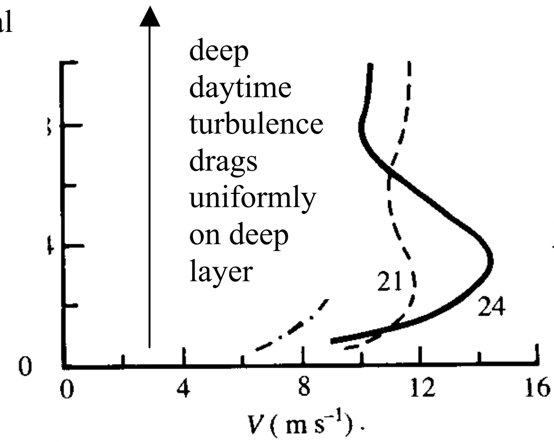
Atmospheric boundary layer/nocturnal jet – forced by sudden reduction in frictional drag at night when convection due to surface solar heating cuts off.



Can you estimate the velocity of the inertial oscillation from the size of the inertial circles?

Fig. 8.3. The historic current measurements in the Baltic by Gustafson and Kullenberg (1936), showing oscillations of near-inertial period. The plot is a progressive vector diagram, showing the displacement a particle would have, given the velocity observed at the current meter. The inertial period is 14 hr, and marks are shown 1 day apart. Note the anticyclonic sense of rotation.

An atmospheric inertial oscillation - The nocturnal jet seen by balloon measurements from Wangara, Australian desert (see Gill 9.7)

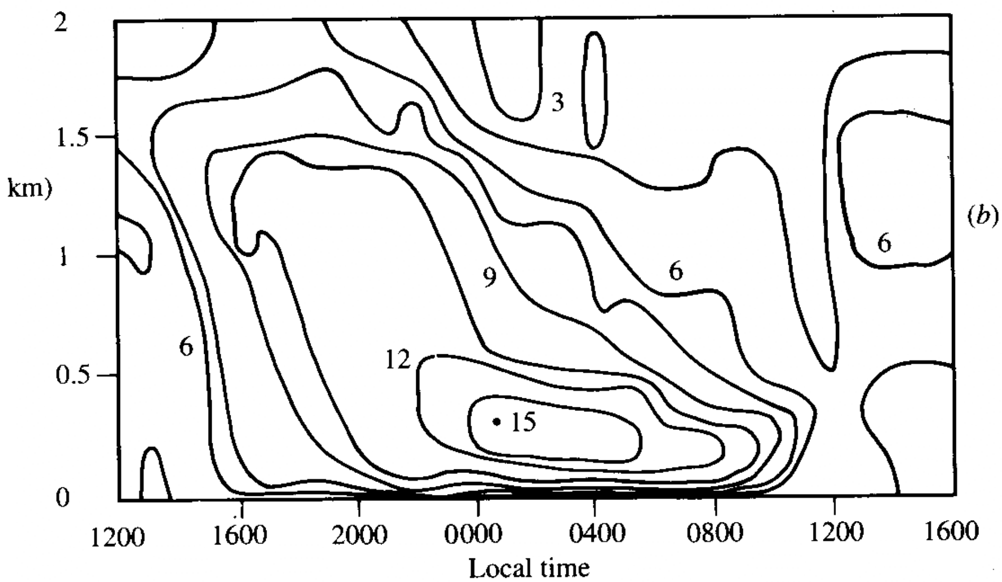


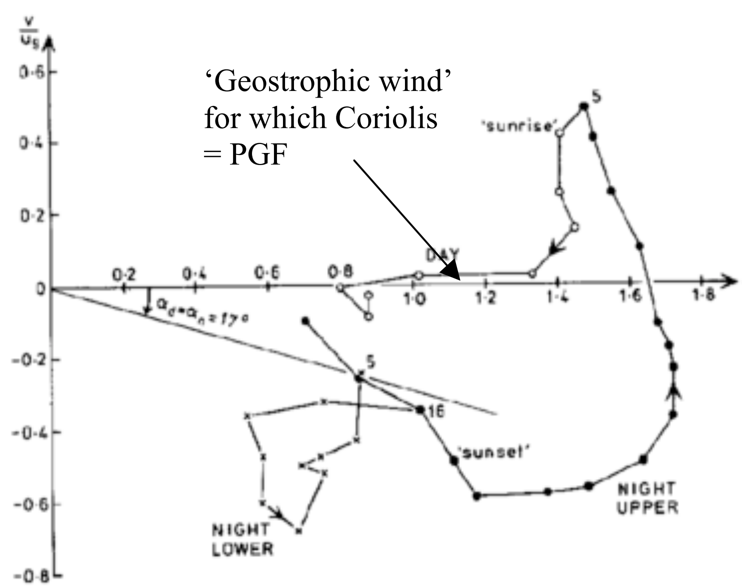
Wind Profiles
(6 pm-midnight)

No drag above

← Nocturnal jet

↑ shallow nighttime turbulence.





200-800 m mean wind turns counterclockwise (why not clockwise?) with time after frictional drag suddenly stops near sunset.

Figure 6. Hodograph of winds averaged over the layers 0–200 m, 200–800 m on day 13/14 of Wangara (27/28 July 1967). Numbers labelling the points refer to Eastern Standard Time.