Important approximations to the equation of motion

1. Boussinesq approximation (buoyancy-driven at all scales in ocean and atmospheric boundary layer)
2. Hydrostatic approximation (flows with much larger horizontal scale than vertical scale)
   a. pressure coordinate formulation of the fluid equations
   b. primitive equations for hydrostatic rotating equation on sphere
3. Geostrophic balance at large scales in ocean and atmosphere

These are derived by scale analysis of the basic fluid equations. For 1 and 2, we will use nonrotating forms of the equations for simplicity; including the Coriolis terms would affect a few details.

Boussinesq approximation (for ocean, atm. bound. layer, lab expts, ideal models)
(6.6.4, but scaling not discussed)

Idea: Sometimes we can treat a compressible fluid as approximately incompressible. Formally, this is the Boussinesq approximation, which consists of:

(1) Neglecting density variations $\left(\frac{1}{\rho}\frac{D\rho}{Dt}\right)$ in the mass continuity equation to get

$$\nabla\cdot\vec{u} = 0 .$$

(filters out sound waves by effectively making $c_s = \infty$)

(2) Linearizing density variations in the momentum equation:

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_{00}}\nabla p' + B\hat{k} \quad , \quad B = -g\,\frac{\rho_\theta - \rho_{00}}{\rho_{00}} \quad ,$$

p' is a pressure perturbation. ρ_{00} is a constant reference density. ρ_θ is potl density.

Validity: We will show the Boussinesq approximation is valid if

(i) Density variations throughout the fluid are small (< 20%, say), so there is a reference density ρ_{00} such that $\frac{\rho-\rho_{00}}{\rho_{00}} \ll 1$ everywhere.

(ii) The generalized Mach number L/c_sT is small, where L is the lengthscale and T the timescale of the flow ($\le L/U$)

Unlike hydrostatic approx does not require $H \ll L$, so good for turbulence.

Derivation (Spiegel and Veronis 1960 Astrophysical J., 131, 442-447)

The derivation of (2) follows the initial steps of our derivation of the hydrostatic approx, but with a constant ρ_0 that may differ from the mean fluid density at any particular height z. This uses only assumption (i).

To derive (1), we must show $\left[\frac{1}{\rho}\frac{D\rho}{Dt}\right]$ is much less than the individual terms $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}$ of $\nabla\cdot\vec{u}$, so the dominant balance in the mass continuity eqn. is of those individual terms with each other.

Let U be hor. velocity scale, L and H the horizontal/vertical lengthscales.

Then
$$\left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}\right] = \frac{U}{L} \quad , \quad [w] = U\frac{H}{L} = W$$

Now, for an adiabatic flow in which entropy η is conserved following fluid parcels,

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_\eta \frac{Dp}{Dt} = \frac{1}{\rho c_s^2}\frac{Dp}{Dt}$$

Partition $p = p_0(z) + p'(x,y,z,t)$ into hydrostatic part $p_0 = p_{00} - \rho_0 g z$ and perturbation p'.

Then

$$\left[\frac{Dp_0}{Dt}\right] = \left[w\frac{\partial p_0}{\partial z}\right] = U\frac{H}{L}\rho_0 g$$

$$\left[\frac{Dp'}{Dt}\right] = \frac{[p']}{T} = \frac{\rho_{00} U L}{T^2}$$ (using horizontal mom eqn $\frac{Du'}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$ to scale p')

Thus

$$\left[\frac{1}{\rho c_s^2}\frac{Dp_0}{Dt}\right] = \left[\frac{1}{\rho c_s^2}\cdot w\cdot \underbrace{\frac{dp_0}{dz}}_{-\rho_0 g}\right] = \frac{gW}{c_s^2} = \frac{W}{H_\eta}$$ (H_η is adiabatic density scale height).

Since we have assumed small density variations throughout fluid (in particular in the vertical) we are obliged to restrict $\boxed{H \ll H_\eta}$

Hence

$$\left[\frac{1}{\rho c_s^2}\frac{Dp_0}{Dt}\right] \ll \left[\frac{\partial w}{\partial z}\right] = \frac{W}{H}$$

Also

$$\left[\frac{1}{\rho c_s^2}\frac{Dp'}{Dt}\right] = \frac{[p']}{\rho_{00} c_s^2 T} = \frac{\rho_{00} U L}{\rho_{00} c_s^2 T^2} = \frac{U}{L}\cdot\left(\frac{L}{c_s T}\right)^2$$

Since the generalized Mach number $L/c_s T \ll 1$,

$$\left[\frac{1}{\rho c_s^2}\frac{Dp'}{Dt}\right] \ll \left[\frac{\partial u}{\partial x}\right] = \frac{U}{L}$$

Consequently, both parts of $\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\rho c_s^2}\frac{Dp}{Dt}$ are small compared to terms in the divergence, so the dominant balance is

$$\nabla\cdot\vec{u} = \nabla_h \cdot \vec{u}_h + \frac{\partial w}{\partial z} = 0$$ (incompressible)

Proof of (2) (Boussinesq simplification of momentum equation)

We divide the density and pressure into hydrostatic reference profiles which depend only on height, plus much smaller perturbations that depend on space and time:

$$p = p_0(z) + p'(x,y,z,t)$$
$$\rho = \rho_0(z) + \rho'(x,y,z,t)$$

Then the pressure gradient and gravity terms on the RHS of the momentum equation can be written:

$$-\frac{\nabla p}{\rho} - g\mathbf{k} = -\frac{\nabla(p_0 + p')}{\rho_0 + \rho'} - g\mathbf{k} = -\frac{\nabla p_0}{\rho_0 + \rho'} - \frac{\nabla p'}{\rho_0 + \rho'} - g\mathbf{k}$$

We assume that $\rho' << \rho_0(z) \approx \rho_{00}$. Then, working on the individual terms,

$$-\frac{\nabla p_0}{\rho_0 + \rho'} = -\frac{-\rho_0 \mathbf{g}}{\rho_0\left(1 + \frac{\rho'}{\rho_0}\right)} \approx \mathbf{g}\left(1 - \frac{\rho'}{\rho_0}\right)$$

and

$$-\frac{\nabla p'}{\rho_0 + \rho'} \approx -\frac{\nabla p'}{\rho_0} \approx -\frac{\nabla p'}{\rho_{00}}$$

Putting these approximations together,

$$-\frac{\nabla p}{\rho} - g\mathbf{k} \approx b\mathbf{k} = -\frac{\nabla p'}{\rho_{00}} + b\mathbf{k}, \quad b = -g\frac{\rho'}{\rho_0}$$

Also we can simplify $b = -g\frac{\rho'}{\rho_0(z)}$, using potential density:

$$\frac{\rho'_\theta}{\rho_{\theta 0}} = \frac{\rho'}{\rho_0} - \frac{1}{\rho_0 c_s^2} p'$$

By scaling the vertical momentum equation, $[p'] = [\rho']gH$ for a density-driven flow, so

$$\frac{[p'/\rho_0 c_s^2]}{[\rho'/\rho_0]} = \frac{gH}{c_s^2} = \frac{H}{H_\eta} \ll 1$$

Using an adiabatic reference profile with potential density $\rho_{\theta 0}$ ($\rho_{\theta 0} = \rho_{00}$) we conclude

$$b = -g\frac{\rho'}{\rho_0(z)} \approx -g\frac{\rho'_\theta}{\rho_{\theta 0}} = -g\left(\frac{\rho_\theta - \rho_{\theta 0}}{\rho_{\theta 0}}\right) = B$$, the Boussinesq buoyancy

Thus we've now demonstrated the second part of the Boussinesq approx.