Hydrostatic Fluid motions on a sphere (Gill 4.12).

Coordinates: longitude λ, latitude φ, height z relative to geoid.

... oblate spheroidal coord. system, but can treat as spherical to a good approximation, with radial distance r from earth center equal to r_0.

Orthogonal coord system with metrics $h_\lambda = r_0\cos\varphi$, $h_\varphi = r_0$, $h_z = 1$, $d\vec{x} = h_\lambda d\lambda\,\hat{e}_\lambda + h_\varphi d\varphi\,\hat{e}_\varphi + h_z dz\,\hat{k}$.

Then...

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r_0\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{r_0}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial z}$$

and

$$\nabla\cdot\vec{F} = \frac{1}{r_0\cos\varphi}\left[\frac{\partial F_\lambda}{\partial\lambda} + \frac{\partial}{\partial\varphi}(F_\varphi\cos\varphi)\right] + \frac{\partial F_z}{\partial z}$$

$\left(u = r_0\cos\varphi\frac{D\lambda}{Dt},\ v = r_0\frac{D\varphi}{Dt},\ w = \frac{Dz}{Dt}\right)$

$r \simeq r_0$

Mass continuity

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{r_0\cos\varphi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\varphi}(v\cos\varphi)\right] + \frac{\partial w}{\partial z} = 0$$

Momentum

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r_0\cos\varphi}\right)(v\sin\varphi - w\cos\varphi) = -\frac{1}{\rho r_0\cos\varphi}\frac{\partial p}{\partial\lambda} + F_u$$

$$\left(\text{or } \frac{Dl}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial\lambda} + r_0\cos\varphi\cdot F_u,\quad l = ur_0\cos\varphi + \Omega r_0^2\cos^2\varphi\right)$$

= angular momentum per unit mass

$$\frac{Dv}{Dt} + \frac{wv}{r_0} + \left(2\Omega + \frac{u}{r_0\cos\varphi}\right)u\sin\varphi = -\frac{1}{\rho r_0}\frac{\partial p}{\partial\varphi} + F_v$$

$$\frac{Dw}{Dt} - \frac{v^2}{r_0} - \left(2\Omega + \frac{u}{r_0\cos\varphi}\right)u\cos\varphi = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_w.$$

+ thermodynamic transport eqns.

Common approximations:

If $\frac{H}{L} \ll 1$ (and under some other circumstances), we can still derive hydrostatic approx, and $\frac{[w\cos\varphi]}{[v\sin\varphi]} \le \frac{[w]}{[v]} = \frac{H}{L} \ll 1$

while $\dfrac{\left[\frac{wv}{r_0}\right]}{\left[\frac{u^2\sin\varphi}{r_0\cos\varphi}\right]} = \dfrac{[wv]}{[v^2]} = \dfrac{H}{L} \ll 1$, so we can neglect terms proportional to w in vert. mom. eqn too.

Assuming $[u] \ll \Omega r_0 \sim 500\,\frac{m}{s}$, can also neglect the terms $\frac{u}{r_0\cos\varphi}$ relative to 2Ω, yielding ('traditional' approximation

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho r_0\cos\varphi}\frac{\partial p}{\partial\lambda} + F_u$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho r_0}\frac{\partial p}{\partial\varphi} + F_v$$

$$0 = -\frac{\partial p'}{\partial z} - \rho' g$$

$f = 2\Omega\sin\varphi$

f-plane

Local Cartesian coords, $dx = r_0 \cos\varphi \, d\lambda$, $dy = r_0 \, d\varphi$, and $f \approx f_0 = 2\Omega \sin\varphi_0$

$$\frac{Du}{Dt} - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_u$$

$$\frac{Dv}{Dt} + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_v$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\nabla_h \cdot \vec{u}_h = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

good if $\frac{L}{r_0} \ll 1$

β-plane

Same, but $f_0 \leftarrow \cancel{f(y)}\, f_0 + \beta y$, where $\beta = \frac{1}{r_0}\frac{df}{d\varphi}\Big|_0 = \frac{2\Omega\cos\varphi_0}{r_0}$, good for somewhat larger scales or lower frequencies. Typical $\beta \sim 1.2\times 10^{-11}\ m^{-1} s^{-1}$, max at equator, zero at poles.

Equatorial β-plane

Center β-plane on equator, so $f(y) = \beta y$.

Geostrophic balance in the traditional approximation (Gill 7.6)

If $Ro = \frac{U}{fL} \ll 1$ and F_u, F_v are also much smaller than Coriolis force the dominant balance in the momentum eqns is:

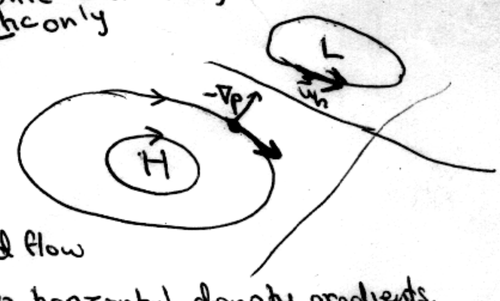
$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

(Geostrophic balance)
- diagnostic only

or

$$f\hat{k}\times\vec{u}_h = -\frac{1}{\rho}\nabla_h p$$



Thermal wind eqn (Gill 7.7)

For a geostrophically, hydrostatically balanced flow we can relate vertical shear in the flow to horizontal density gradients.

$$f\hat{k}\times\underbrace{\frac{1}{\rho}\frac{\partial}{\partial z}[\rho\vec{u}_h]}_{\vec{S}\text{, the shear vector}} = g\,\frac{\nabla_h\rho}{\rho}$$

$\rho' > 0$ (dense)

$\nabla\rho$ $\vec{S}$

$\rho' < 0$ (light)

p-coord form

$$f\hat{k}\times\vec{u}_h = -\nabla_h\Phi$$

$$f\hat{k}\times\vec{S} = -\rho g\,\nabla_h\!\left(\frac{\partial\Phi}{\partial p}\right) = g\frac{\nabla_h\rho}{\rho}, \quad \vec{S} = -\rho g\frac{\partial\vec{u}_h}{\partial p}$$

Boussinesq

$$f\hat{k}\times\vec{u}_h = -\frac{1}{\rho_{00}}\nabla p'$$

$$f\hat{k}\times\vec{S} = -\nabla_h b$$

$$\vec{S} = \frac{\partial\vec{u}_h}{\partial z}.$$