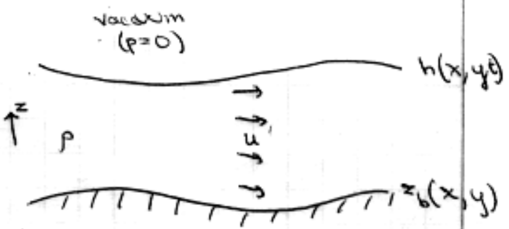
Shallow Water Equations (CR 4.2-3; Gill 5.6)

Basis: Hydrostatic approx : $p(x,y,t) = \rho g\left(h(x,y,t) - z\right)$



$\Rightarrow \frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x}, \quad \frac{\partial p}{\partial y} = \rho g \frac{\partial h}{\partial y}$ ind. of z

$\Rightarrow$ Consistent to assume u, v ind. of z

No friction,

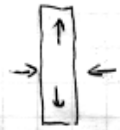
$$\frac{Du}{Dt} - fv = -g\frac{\partial h}{\partial x} \qquad (1)$$

$$\frac{Dv}{Dt} + fu = -g\frac{\partial h}{\partial y} \qquad (2)$$

$$\frac{D}{Dt}(h - z_b) = w(h) - w(z_b) = (h - z_b)\frac{\partial w}{\partial z} = -(h - z_b)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (3)$$

Flat bottom:

$$\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$



Low Ro flow (CR 4.1)

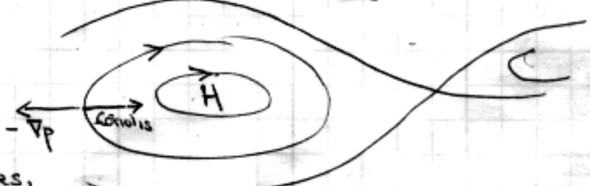
Let $[u,v] = U$, $[x,y] = L$, $[t] = T = L/U$. Then $\frac{[Du/Dt]}{[fv]} = \frac{U^2/L}{fU} = \frac{U}{fL} = Ro$

If $Ro \ll 1$,

$$-fv_g \approx -g\frac{\partial h}{\partial x} = -fv_g \qquad \text{(geostrophic balance)}$$

$$fu \approx -g\frac{\partial h}{\partial y} = fu_g,$$ In fact $\frac{[u-u_g]}{[u]} = Ro$.

Coriolis and pressure gradient accelerations balance.



Note this is diagnostic, this leading balance cannot predict changes in flow. Also note that if z_B varies, $\left[\frac{D}{Dt}(h - z_B)\right] = O(Ro)\cdot\frac{U}{L} \ll \frac{U}{L}$, since flow nearly nondivergent.

Vorticity + PV (CR 4.4, Gill 7.10)

$\nabla\times\vec{u} \approx \zeta\hat{k}$

Let $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. Then $(1\text{-}2) \Rightarrow$

$$\underbrace{\frac{\partial}{\partial x}\frac{Dv}{Dt} - \frac{\partial}{\partial y}\frac{Du}{Dt}}_{I} + \underbrace{v\frac{\partial f}{\partial y}}_{Df/Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$I = \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right\} - \frac{\partial}{\partial y}\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\}$$

$$= \frac{\partial \zeta}{\partial t} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} + v\frac{\partial}{\partial y}\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} = \frac{D\zeta}{Dt} + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\Rightarrow \quad 0 = \frac{D(\zeta+f)}{Dt} + (\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \qquad \text{Absolute vorticity eqn}$$

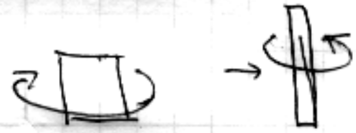
or $$\frac{1}{\zeta+f}\frac{D}{Dt}(\zeta+f) = \frac{\partial w}{\partial z}$$ ... interpret in terms of vortex stretching.

Now $$\frac{1}{h-z_b}\frac{D}{Dt}(h-z_b) = \frac{\partial w}{\partial z}$$, too, so

$$\frac{1}{\zeta+f}\frac{D}{Dt}(\zeta+f) - \frac{1}{h-z_b}\frac{D}{Dt}(h-z_b) = 0$$

$$\frac{D}{Dt}\ln q = 0 \quad , \quad q = \frac{\zeta+f}{h-z_b} = \text{potential vorticity.}$$

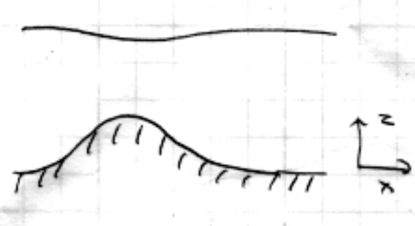
Thus q is conserved following fluid columns.



Note:

$$Ro \ll 1 \Rightarrow \frac{[\zeta]}{f} = \frac{U}{fL} \ll 1 .$$

Since q is conserved by columns, on an f-plane, $h - z_b = q\cdot(\zeta+f)$ can only vary by $O(Ro)$. ~~Since~~ Thus fluid flow must lie approximately along lines of constant fluid depth. If the surface is much flatter than the bottom, fluid will be trapped over bumps or dips as Taylor columns. For topography of scale L, the height perturbation at the fluid surface scale as



SWE $\quad g\frac{[h]}{L} = fU \Rightarrow [h] = \frac{fUL}{g}$

Thus, Taylor columns will form over topography with greater undulation than this,

if $\quad [z_b] \gg \frac{fUL}{g} = \frac{f^2L^2}{g}\cdot Ro .$