Linearized SWE for infinitesimal disturbances

$z_B,\ u,\ v,\ \eta = h - H \ll 1$

Then all products of small quantities are neglected, so

$$\frac{\partial u}{\partial t} + \cancel{(\vec{u}\cdot\nabla)\vec{u}} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \cancel{(\vec{u}\cdot\nabla)\vec{v}}^{\,0} + fu = -g\frac{\partial \eta}{\partial y} \qquad \text{(LSWE)}$$

$$\frac{\partial}{\partial t}(\eta - \cancel{z_b}^{\,0}) + H\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0\,.$$

$$\cancel{+(\vec{u}\cdot\nabla)(\eta - z_b)} \quad \cancel{+(\eta - z_b)\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]}$$

We'll see later that with appropriate redefinition of u, g and H, the exact same equations apply to systems with two or more layers with different densities. For a two layer system with $\rho_2 - \rho_1 \ll \rho_1$, we'll see:

$$g_{eff} = g\frac{\rho_2 - \rho_1}{\rho_2} \quad (\ll g\,!)$$

$$H_{eff} = \frac{H_1 H_2}{H_1 + H_2}$$

$$u_{eff} = u_2 - u_1\,.$$

Poincaré Waves (CR6.3)

Strategy: Look for sinusoidal disturbances to the LSWE in a horizontally unbounded domain on an f-plane. All coeffs of LSWE constant (ind. of x,y,t) so this can work (represents Fourier transform of LSWE in x,y,t)

$$\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \mathrm{Re}\left\{\begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\eta} \end{pmatrix} e^{i(kx + ly - \omega t)}\right\}. \qquad \begin{aligned}(k,l) &= \text{hor. wavenumber} \\ \omega &= \text{frequency}\end{aligned}$$

When we find a soln we take its real part to obtain a "physical" solution.

Note
$$\frac{\partial}{\partial x} \leftarrow ik\,,\quad \frac{\partial}{\partial y} \leftarrow il\,,\quad \frac{\partial}{\partial t} \leftarrow -i\omega$$

$$-i\omega\hat{u} - f\hat{v} = -g\cdot ik\hat{\eta} \qquad (1)$$

$$-i\omega\hat{v} + f\hat{u} = -g\cdot il\hat{\eta} \qquad (2)$$

$$-i\omega\hat{\eta} + H_0\{ik\hat{u} + il\hat{v}\} = 0. \qquad (3)$$

Eliminate $\hat{u}, \hat{v}$ in terms of $\hat{\eta}$ using (1) + (2)

$$\hat{u} = \frac{\omega k + ifl}{\omega^2 - f^2}\, g\hat{\eta}$$

$$\hat{v} = \frac{\omega l - ifk}{\omega^2 - f^2}\, g\hat{\eta} \qquad \text{(Polarization relations)}$$

Plug into (3)

$$-i\omega\left\{\omega^2 - f^2 - gH_0(k^2 + l^2)\right\}\hat{\eta} = 0. \qquad \text{(Dispersion relation)}$$

$$\Rightarrow \omega = 0,\quad \hat{u} = -\frac{ilg\hat{\eta}}{f} \Leftrightarrow u = -\frac{g}{f}\frac{\partial \eta}{\partial y} = u_g\,,\quad \hat{v} = \frac{ikg\hat{\eta}}{f} \Leftrightarrow v = \frac{g}{f}\frac{\partial \eta}{\partial x} = v_g \quad \text{(Geostrophic mode)}$$

or $\omega^2 = f^2 + c_0^2\underbrace{(k^2+l^2)}_{\kappa^2}$, $c_0 = (gH_0)^{\frac{1}{2}}$

Note rotation only significant if $\kappa \lesssim \frac{1}{R}$, $R = \frac{c_0}{f}$ = Rossby radius;

if we'd used full eqn's,

$$-i\omega\left(\omega^2 - f^2 - gH_0(k^2+l^2)\right)\begin{pmatrix}\eta\\u\\v\end{pmatrix} = 0.$$

$$\text{or } \frac{\partial}{\partial t}\left(-\frac{\partial^2}{\partial t^2} - f^2 + gH_0\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right)\begin{pmatrix}\eta\\u\\v\end{pmatrix} = 0.$$

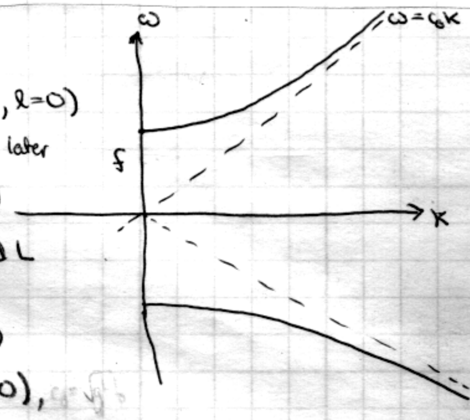
This is useful for finite-domain problems.

Lab ($f = 2\Omega = 6\,s^{-1}$, $H = 0.1\,m$, $g = 10\,m/s^2$ ⇒ $c_0 = 1\frac{m}{s}$, $R = 16cm$; $g' = 0.4\frac{m}{s}$ ⇒ $c_0 = 0.2\frac{m}{s}$, $R = 3\,cm$.)

Ocean $H_{eff} = 1\,km$ ($H_1 = 1, H_2 = \infty$), $g' = 0.03\frac{m}{s^2}$ ⇒ $c_0 = 5\frac{m}{s}$, $R = 50km$

Atm $H_{eff} = 2.5km$ ($H_1 = H_2 = 5$), $g_{eff} = 0.7\frac{m}{s}$ ⇒ $c_0 = 40\frac{m}{s}$, $R = 400km$.

Dispersion Diagram



Later For a wave propagating in the +x direction ($\kappa = k$, $l = 0$)

$c_{ph} = \frac{\omega}{k} > c_0 = \sqrt{gH_0}$, $\left[c_g = \frac{\partial\omega}{\partial k} = \frac{c_0^2}{c_{ph}} < c_0\right]$ later

$= c_0\left[1+\frac{f^2}{c_0^2k^2}\right]^{\frac{1}{2}}$

Note that $\omega > 0$ and $\omega < 0$ correspond to R and L moving waves. $f = 0$ case - nondispersive c_{ph} ind. of k.

For a R-moving wave with $k > 0$, $l = 0$, $\omega > 0$,

$$\hat{u} = \frac{k\omega g}{\omega^2 - f^2}\hat{\eta} = \frac{g}{c_0}\hat{\eta} \quad (f=0),$$

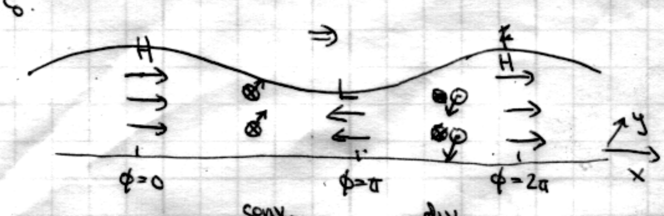
$$\hat{v} = \frac{-ikfg}{\omega^2 - f^2}\hat{\eta} = 0 \quad (f=0)$$

i.e. $\eta(x,t) = \text{Re}\ \hat{\eta}e^{i(kx+ly-\omega t)} = \eta_0\cos(\overbrace{kx - \omega t}^{\phi})$ $\hat{\eta} = \eta_0$ real,

$u(x,y) = \text{Re}\ \hat{u}e^{i(kx+ly-\omega t)} = \frac{kg\omega}{\omega^2-f^2}\eta_0\cos(kx - \omega t)$ (in phase with η)

$v(x,y) = \text{Re}\ \frac{-ikfg}{\omega^2-f^2}\eta_0 e^{i(kx+ly-\omega t)} = \frac{kfg}{\omega^2-f^2}\eta_0\sin(kx - \omega t)$

Note if $f = 0$, $v = 0$; and $c_{ph} = c_0$ for all k's.
(gravity wave)



At a given x, ϕ decreases at a rate $\frac{\partial\phi}{\partial t} = -\omega$, so we can scroll from R to L to see behavior at one position at a succession of times. Viewed this way, we see the column velocity changing due to coriolis turning + pressure gradients, and column height responding to div/conv.

At $\phi = \frac{\pi}{2}$, $\zeta = \frac{\partial v}{\partial x} = 0$ and $\eta = 0$ so $q = \frac{\zeta + f}{\eta + H} = f/H$ ⇔ wave carries no PV perturbation.

If $k \to 0$, $\omega^2 - f^2 = c_0^2k^2 \to 0$ so $\omega \approx \pm f$. These are inertial oscillations!

u and v become large w.r.t. η_0, and $|\hat{u}| = |\hat{v}| = \frac{gf}{c_0^2 k}\eta_0$.

Note steady geostrophic modes with $\omega = 0$ too, geostrophic + carry the PV in the flow.