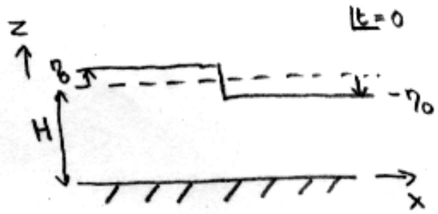
LSWE Dambreak problem (Gill 5.6 ($f=0$); 7.2 ($f\neq 0$))

$$\eta(x,t=0)=\begin{cases}\eta_0 & x<0\\ -\eta_0 & x>0\end{cases}$$

$$u(x,0)=0$$

$$v(x,0)=0$$



LSWE:
$$u_t - fv = -g\eta_x$$
$$v_t + fu = 0 \quad\Rightarrow\quad \eta_{tt}-c_0^2\eta_{xx}+f^2\eta=-f^2\eta_0\,\mathrm{sgn}\,x$$
$$\eta_t + Hu_x = 0$$

Nonrotating case ($f=0$) $\quad \eta_{tt}-c_0^2\eta_{xx}=0$ (wave eqn)

$$\Rightarrow \eta = F(x-c_0t)+G(x+c_0t)$$

$$u_t=-g\eta_x=-g\left[F'(x-c_0t)+G'(x+c_0t)\right]$$

$$\Rightarrow u=\frac{g}{c_0}\left[F(x-c_0t)-G(x-c_0t)\right]$$

At $t=0$

$$0=u(x,0)=\frac{g}{c_0}\left[F(x)-G(x)\right]\Rightarrow G(x)=F(x)$$

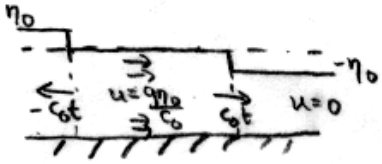
$$\eta(x,0)=F(x)+G(x)=2F(x)=\begin{cases}\eta_0 & x<0\\ -\eta_0 & x>0\end{cases}=-\eta_0\,\mathrm{sgn}(x)$$

$$\Rightarrow F(x)=-\frac{\eta_0}{2}\,\mathrm{sgn}(x)=G(x)$$

$f=0$, $t>0$

$$u(x,t)=-\frac{g\eta_0}{2c_0}\left\{\mathrm{sgn}(x-c_0t)-\mathrm{sgn}(x+c_0t)\right\}$$

$$\eta(x,t)=-\frac{\eta_0}{2}\left\{\mathrm{sgn}(x-c_0t)+\mathrm{sgn}(x+c_0t)\right\}$$



Rotating case ... has a similarity solution

$$u=\begin{cases}\dfrac{g\eta_0}{c_0}J_0(\tau), & \tau=f\left(t^2-\dfrac{x^2}{c_0^2}\right)^{\frac{1}{2}}, \text{ for } |x|<c_0t\\ 0, & |x|>c_0t\end{cases}$$

v, η derived from u.

... sharp wavefronts at $x=\pm c_0t$ with tail of longer Poincare wave lagging behind (since $c_g<c_0$ for longer waves), leaving geostrophic steadystate.

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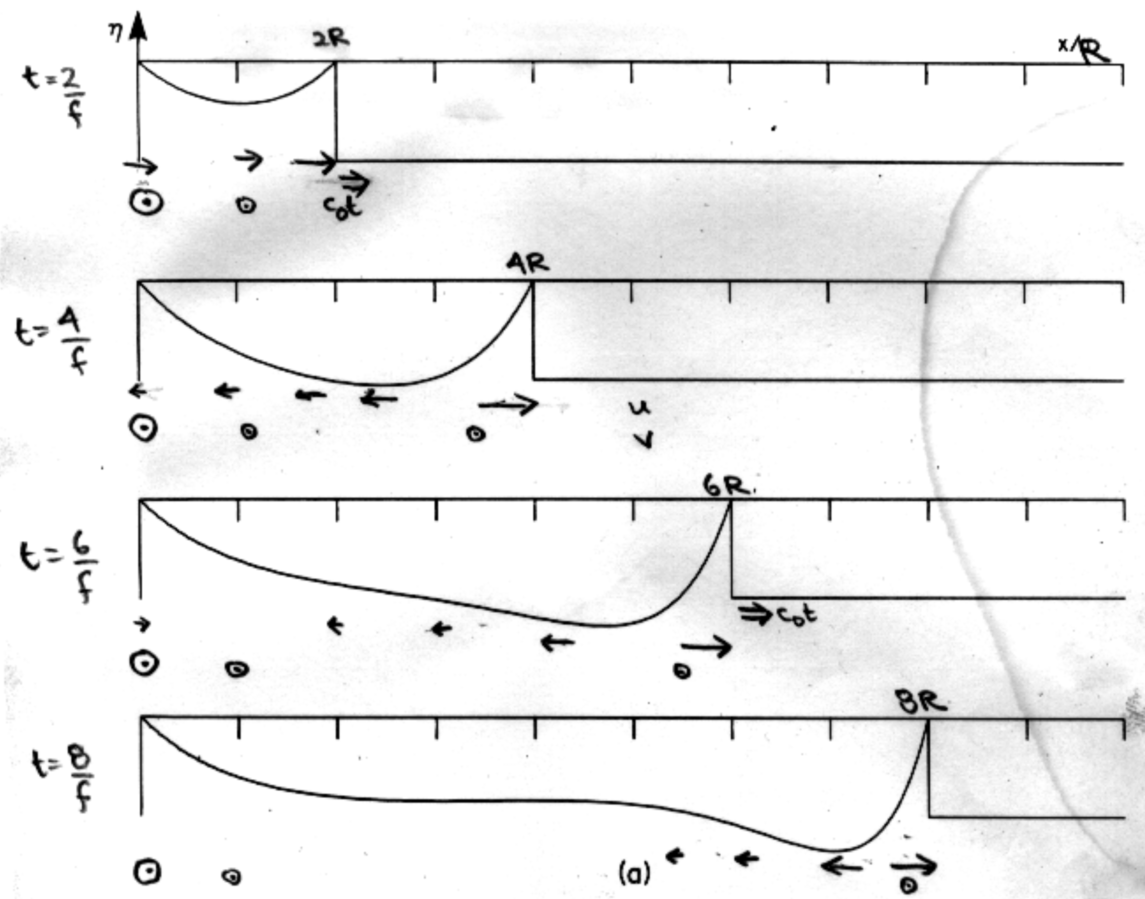

Fig. 7.3. Transient profiles for (a) η, (b) u, and (c) v for adjustment under gravity of a fluid with ar finitesimal discontinuity in level of $2\eta_0$ at $x = 0$. The solution is shown in the region $x > 0$, where the su initially depressed, at time intervals of $2f^{-1}$, where f is twice the rate of rotation of the system about a ve The marks on the x axis are at intervals of a Rossby radius, i.e., $(gH)^{1/2}/f$, where g is the acceleration due and H is the depth of fluid. The solutions retain their initial values until the arrival of a wave front that t from the position of the initial discontinuity at speed $(gH)^{1/2}$. When the front arrives, the surface elevatic η_0 and the u component of velocity rises by $(g/H)^{1/2}\eta_0$ just as in the nonrotating case depicted in Fig. 5 because the first waves to arrive are the very short waves, which are unaffected by rotation. Behind however, is a "wake" of waves produced by dispersion, which in the case of u, have the slope given by function (7.3.14). This is the point impulse solution to the Klein–Gordon equation. The "width" of the fro in inverse proportion with time. Well behind the front, the solution adjusts to the geostrophic equilibriu depicted in Fig. 7.1.

As $t \to \infty$ at fixed x, the Poincaré wavetrain passes through, leaving behind a geostrophically balanced steady-state solution

Steady state LSWE $\Rightarrow$ (no y dependence)

$$\overset{0}{\cancel{u_t}} - fv = -g\eta_x \quad \Rightarrow \text{geostrophic balance in } x$$

$$\overset{0}{\cancel{v_t}} - fu = 0 \quad \Rightarrow u = 0 \ \left(\text{geostrophic balance in } y, \text{ since } \eta_y = 0\right)$$

$$\overset{0}{\cancel{\eta_t}} + H\overset{0}{\cancel{u_x}} = 0 \quad \Rightarrow \text{no new info}$$

We need additional information to determine steady state, which is given to us by PV conservation – each fluid column remembers its initial PV. For the nonlinear SWE, we showed

$$\frac{Dq}{Dt} = 0 \qquad , \; q = \frac{\zeta + f}{\eta + H} \qquad \text{(PV Cons)}$$

For the LSWE, a comparable PV conservation can be derived directly from the linearized eqns of motion (homework) or by linearizing (PV cons), noting $\frac{\zeta}{f}, \frac{\eta}{H} \ll 1$:

$$q = \frac{f\left(1 + \frac{\zeta}{f}\right)}{H\left(1 + \frac{\eta}{H}\right)} \approx \frac{f}{H}\left\{1 + \frac{\zeta}{f} - \frac{\eta}{H}\right\} = q_0 + q'(x,t)$$

where $q_0 = \frac{f}{H}$ (constant), $q' = \frac{f}{H}\left\{\frac{\zeta}{f} - \frac{\eta}{H}\right\}$

Thus

$$0 = \frac{Dq}{Dt} = \left\{\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \overset{0}{\cancel{v\frac{\partial}{\partial y}}}\right\}(q_0 + q') = \frac{\partial q'}{\partial t} + \cancel{u\frac{\partial q'}{\partial x}}$$

neglect since (small)^2

$$\Rightarrow \boxed{\frac{\partial q'}{\partial t} \propto \frac{\partial}{\partial t}\left\{\frac{\zeta}{f} - \frac{\eta}{H}\right\} = 0 \text{. (LinPVCons)}}$$

... the perturbation PV is conserved in quasi-stationary fluid columns.

$\frac{\eta}{H}$ increases $\Rightarrow$ column stretches $\Rightarrow$ spins up $\Rightarrow$ $\frac{\zeta}{f}$ increases.

Application to dambreak:

$$\frac{\zeta(x,t)}{f} - \frac{\eta(x,t)}{H} = \overset{0}{\cancel{\frac{\zeta(x,0)}{f}}} - \frac{\eta(x,0)}{H} = \frac{\eta_0}{H}\operatorname{sgn}(x) \quad , \; \zeta = \frac{\partial v}{\partial x} - \overset{0}{\cancel{\frac{\partial u}{\partial y}}}$$

Combine with steady state geostrophic balance: $v = \frac{g\eta_x}{f}$

$$\Rightarrow \frac{g\eta_{xx}}{f^2} - \frac{\eta}{H} = \frac{\eta_0}{H}\operatorname{sgn}(x) \quad \Rightarrow R^2\eta_{xx} - \eta = \eta_0 \operatorname{sgn}(x), \; R = \frac{c_0}{f}$$

= Rossby radius

With BC $\eta \to -\eta_0 \operatorname{sgn}(x)$ as $x \to \pm\infty$, the solution is

$$\eta = -\eta_0 \operatorname{sgn}(x)\left\{1 - e^{-|x|/R}\right\}$$

$$v = -\frac{g\eta_0}{c_0} e^{-|x|/R}$$