## General Fourier solution to LSWE on f-plane

$$\begin{bmatrix} u \\ v \\ \eta \end{bmatrix}(x,t) = \underbrace{\int_{-\infty}^{\infty} a^{(r)}(k) \begin{bmatrix} \hat{u}^{(r)} \\ \hat{v}^{(r)} \\ \hat{\eta}^{(r)} \end{bmatrix} e^{i(kx-\omega t)}\, dk}_{\text{right-moving Poincare waves}} + \underbrace{\int_{-\infty}^{\infty} a^{(l)}(k) \begin{bmatrix} \hat{u}^{(l)} \\ \hat{v}^{(l)} \\ \hat{\eta}^{(l)} \end{bmatrix} e^{i(kx+\omega t)}\, dk}_{\text{left-moving waves}}$$

$$+ \underbrace{\int_{-\infty}^{\infty} a^{(g)}(k) \begin{bmatrix} \hat{u}^{(g)} \\ \hat{v}^{(g)} \\ \hat{\eta}^{(g)} \end{bmatrix} e^{ikx}\, dk}_{\text{steady geostrophic modes}}, \quad \omega = k c_p(k), \quad c_p = \left(gH + \frac{f^2}{k^2}\right)^{\frac{1}{2}}$$

Here: $a^{(r,l,g)}$ are the amplitudes of the three modes

WLOG, $\hat{\eta}^{(r,l,g)} = H$ (modal amplitudes defined in terms of nondim surf height perturbations, which are small)

From polarization relns with $l = 0$:

$$\hat{u}^{(r)} = \frac{\omega k + \cancel{ifl}}{\underbrace{\omega^2 - f^2}_{gHk^2}} \cdot g\underbrace{\hat{\eta}^{(r)}}_{H} = \frac{\omega}{k} = c_p \;; \quad \hat{v}^{(r)} = \frac{\cancel{\omega l} - ifk}{\underbrace{\omega^2 - f^2}_{gHk^2}} \cdot gH = -\frac{if}{k}$$

$\omega \leftarrow -\omega$

$$\hat{u}^{(l)} = -\frac{\omega}{k} = -c_p \;; \quad \hat{v}^{(l)} = \frac{\cancel{-\omega l} - ifk}{\omega^2 - f^2} \cdot gH = -\frac{if}{k}$$

$\omega \leftarrow 0$

$$\hat{u}^{(g)} = \frac{ifl}{-f^2} \cdot gH = 0 \;; \quad \hat{v}^{(g)} = \frac{-ifk}{-f^2} \cdot gH = \frac{ik}{f} \cdot gH$$

Measure $x$ in units of $R = \frac{\sqrt{gH}}{f}$ and $t$ in units of $f$, so

$gH \leftarrow 1$, $f \leftarrow 1$:

$$c_p(k) = \left(1 + \frac{1}{k^2}\right)^{\frac{1}{2}}$$

$$\hat{u}^{(r)} = c_p, \quad \hat{v}^{(r)} = -\frac{i}{k}, \quad \hat{\eta}^{(r)} = 1$$

$$\hat{u}^{(l)} = -c_p, \quad \hat{v}^{(l)} = -\frac{i}{k}, \quad \hat{\eta}^{(l)} = 1$$

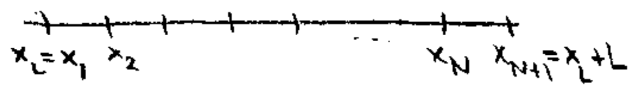
$$\hat{u}^{(g)} = 0, \quad \hat{v}^{(g)} = ik, \quad \hat{\eta}^{(g)} = 1$$

## Numerical discretization of the Fourier representation

The Fourier representation of the general soln to the LSWE is very practical for numerically generating solns with arbitrary initial conditions

For this purpose, we usually consider a soln. generated from an initial grid of N uniformly-spaced points spanning a periodic domain of some length L:

$$x_j = x_L + (j-1)\Delta x, \quad \Delta x = \frac{L}{N} \quad , j = 1, .., N \quad , x_L \text{ arbitrary}$$




There are N Fourier modes that can be unambiguously represented on this points, which we index $m = 1, .., N$

Each corresponds to a "harmonic" $M_m$ with wavenumber

$$k_m = \frac{2\pi M_m}{L}$$

For use with the Discrete Fourier Transform (DFT) they are interpreted as follows

| m | 1 | 2 . . . | $\frac{N}{2}$ | $\frac{N}{2}+1$ | . . . . | N |
|---|---|---|---|---|---|---|
| M | 0 | 1 . . . | $\frac{N}{2}-1$ | $-\frac{N}{2}$ | . . . . | $-1$ |
| | constant | positive harmonics | | negative harmonics | | |

Then our Fourier soln, restricted to these wavenumbers is:

$$\begin{bmatrix} u \\ v \\ \eta \end{bmatrix}(x,t) = \frac{1}{N}\sum_{m=1}^{N} a_m^{(r)} \begin{bmatrix} \hat{u}_m^{(r)} \\ \hat{v}_m^{(r)} \\ 1 \end{bmatrix} e^{i(k_m x - \omega_m t)}$$

$$+ \frac{1}{N}\sum_{m=1}^{N} a_m^{(l)} \begin{bmatrix} \hat{u}_m^{(l)} \\ \hat{v}_m^{(l)} \\ 1 \end{bmatrix} e^{i(k_m x + \omega_m t)}$$

$$+ \frac{1}{N}\sum_{m=1}^{N} a_m^{(g)} \begin{bmatrix} \hat{u}_m^{(g)} \\ \hat{v}_m^{(g)} \\ 1 \end{bmatrix} e^{i k_m x} \quad , \quad \omega_m = k_m\left(1+\frac{1}{k_m^2}\right)^{\frac{1}{2}}$$

i.e.

$$\vec{u} = \left\{ \begin{matrix} u(x_j,t), \\ j=1,..,N \end{matrix} \right\} = \underbrace{\text{IDFT}}_{\substack{\text{Matlab} \\ \text{ifft}}} \left\{ \begin{matrix} a_m^{(r)}\hat{u}_m^{(r)} e^{-i\omega_m t} \\ + a_m^{(l)}\hat{u}_m^{(l)} e^{i\omega_m t} \\ + a_m^{(g)}\hat{u}_m^{(g)}, \\ m=1,..,N \end{matrix} \right\}$$

Given an initial state $\vec{u}^{(0)}, \vec{v}^{(0)}, \vec{\eta}^{(0)}$, where $\vec{u}_0 = \begin{bmatrix} u(x_1,0) \\ u(x_2,0) \\ \vdots \\ u(x_N,0) \end{bmatrix}$, etc.

we can calculate the corresponding $a_m^{(r,l,g)}$ for the three modes as follows.

$$\vec{\hat{u}}^{(0)} = \text{DFT}\left[\vec{u}^{(0)}\right] \quad , \text{ similarly for } \vec{\hat{v}}^{(0)}, \vec{\hat{\eta}}^{(0)}$$

$$\begin{aligned} \hat{u}_m^{(0)} &= a_m^{(r)}\hat{u}_m^{(r)} + a_m^{(l)}\hat{u}_m^{(l)} + a_m^{(g)}\hat{u}_m^{(g)} \\ \hat{v}_m^{(0)} &= a_m^{(r)}\hat{v}_m^{(r)} + a_m^{(l)}\hat{v}_m^{(l)} + a_m^{(g)}\hat{v}_m^{(g)} \qquad m=1,..,N \\ \hat{\eta}_m^{(0)} &= a_m^{(r)} + a_m^{(l)} + a_m^{(g)} \end{aligned}$$

which is a 3×3 matrix eqn for $\vec{a}_m = \begin{bmatrix} a_m^{(r)} \\ a_m^{(l)} \\ a_m^{(g)} \end{bmatrix}$ given known quantities

$$\begin{bmatrix} \hat{u}_m^{(0)} \\ \hat{v}_m^{(0)} \\ \hat{\eta}_m^{(0)} \end{bmatrix} = \begin{bmatrix} c_p & -c_p & 0 \\ -\frac{i}{k} & -\frac{i}{k} & ik \\ 1 & 1 & 1 \end{bmatrix} \vec{a}_m \Rightarrow \boxed{\begin{aligned} a_m^{(g)} &= \frac{\hat{\eta}_m^{(0)} - ik\hat{v}_m^{(0)}}{1+k^2} \\ a_m^{(r)} &= \frac{1}{2}\left(\hat{\eta}_m^{(0)} + \frac{\hat{u}_m^{(0)}}{c_p}\right) - \frac{a_m^{(g)}}{2} \\ a_m^{(l)} &= \frac{1}{2}\left(\hat{\eta}_m^{(0)} - \frac{\hat{u}_m^{(0)}}{c_p}\right) - \frac{a_m^{(g)}}{2} \end{aligned}}$$