Homework 2 solutions

1. (a) We start with the Boussinesq ensemble-averaged equations for θ, q with reference density ρ_R. The mean advection terms are zero since **u** = 0. Since we also neglect sources of θ, q (S_θ, S_q = 0):

$$\frac{\bar{D}\bar{\theta}}{Dt} = -\frac{\partial}{\partial z}\overline{w'\theta'},$$

$$\frac{\bar{D}\bar{q}}{Dt} = -\frac{\partial}{\partial z}\overline{w'q'},$$

In this problem, since the BL is horizontally homogeneous, ensemble-averaging is equivalent to horizontal averaging. In a mixed layer, $\bar{\theta}$ and $\bar{q}$ do not depend on z, so $\partial\bar{\theta}/\partial t$ and $\partial\bar{q}/\partial t$ are also z-independent. Hence the flux convergences must be z-independent, i. e., $\overline{w'\theta'}$ and $\overline{w'q'}$ are linear functions of z. Hence the buoyancy flux, which is a linear combination of the heat and moisture fluxes, will also be a linear function of z.

(b) $\rho_R c_p \overline{w'\theta'}|_0$ = 300 W m^{-2} => $\overline{w'\theta'}|_0$ = (300 W m^{-2})/[(1.2 kg m^{-3})(10^3 J kg^{-1} K^{-1})]
= 0.25 K m s^{-1}

$\rho_R L \overline{w'q'}|_0$ = 300 W m^{-2} => $\overline{w'q'}|_0$ = (300 W m^{-2})/[(1.2 kg m^{-3})(2.5x10^6 J kg^{-1})]
= 10^{-4} (m s^{-1})(kg kg^{-1})

so $B_0 = g(\overline{w'\theta'}|_0/\theta_{vR} + 0.61\,\overline{w'q'}|_0)$
= (9.8 m s^{-2})[(0.25 K m s^{-1})/(300 K) + 0.61(10^{-4} m s^{-1})]
= 8.8x10^{-3} m^{2}s^{-3}

(c) Ensemble-averaged TKE equation:

$$\partial\bar{e}/\partial t + \bar{\mathbf{u}}\cdot\nabla\bar{e} = S + B + T + D\,.$$

Since there is no mean wind, the vertical shear of $\bar{\mathbf{u}}$ is zero, so advection of TKE by the mean wind is zero. Combining this with assumption (iv) that TKE tendency is zero, the left hand side is negligible. In addition, so

$$S = --\overline{u'w'}\frac{\partial\bar{u}}{\partial z} - \overline{v'w'}\frac{\partial\bar{v}}{\partial z} = 0\,.$$

Integrating the TKE equation over the BL depth:

$$0 = \int_0^h B\,dz + \int_0^h T\,dz + \int_0^h D\,dz$$

Now, using assumptions (i) and (ii), and noting w' = 0 at the surface,

$$\int_0^h T\,dz = -\left[\overline{w'e'} + \rho_R^{-1}\overline{w'p'}\right]_0^h = 0\,.$$

From assumption (iii),

$$\int_0^h D\,dz = h\left(-\frac{V^3}{h}\right) = -V^3 .$$

Thus

$$0 = \int_0^h B\,dz - V^3 \text{, i. e. } V = \left(\int_0^h B\,dz\right)^{1/3} .$$

(d) From (a), the buoyancy flux $B(z)$ is a linear function of z. Since $B(h) = -0.2B_0$,

$$B(z) = B_0(1 - 1.2z/h),$$

$$\int_0^h B\,dz = 0.4B_0 h .$$

Hence

$$V = \{0.4B_0 h\}^{1/3} = \{(0.4)(8.8\text{x}10^{-3}\ \text{m}^2\text{s}^{-3})(1000\ \text{m})\}^{1/3} = \underline{1.5\ \text{m s}^{-1}}$$

and $D = -V^3/h = 0.4B_0 = -\underline{3.5\text{x}10^{-3}}\ \text{m}^2\text{s}^{-3}$. Note that this is proportional to the standard convective velocity scale w_*, but differs from it by a factor of $0.4^{1/3}$.

(e) Substituting the given forms of w' and b' into the definition of buoyancy flux, and recognizing that a horizontal average is just an average over one horizontal wavelength $2h$,

$$\overline{w'b'}\left(\frac{h}{2}\right) = \frac{1}{2h}\int_0^{2h}\left(V\sin\frac{\pi x}{h}\right)\left(\delta b\sin\frac{\pi x}{h}\right)dx = \frac{V\delta b}{2}$$

Thus,

$$\delta b = 2\overline{w'b'}\left(\frac{h}{2}\right)/V = 0.8B_0/V = (0.8)(8.8\text{x}10^{-3}\ \text{m}^2\text{s}^{-3})/(1.5\ \text{m s}^{-1})$$

$$= \underline{4.7\text{x}10^{-3}}\ \text{m s}^{-2}.$$

By assumption, $\overline{w'q'}\ (h/2) = \overline{w'q'}\ |_0 = 10^{-4}$ (m s^{-1})(kg kg^{-1}), so via the above reasoning,

$$\delta q = 2\overline{w'q'}\ (h/2)/V = 1.3\text{x}10^{-4}\ \text{kg kg}^{-1} = \underline{0.13\ \text{g kg}^{-1}}$$

We can back out the midheight θ-flux from the buoyancy and moisture fluxes at $h/2$:

$$\overline{w'\theta'} = \theta_{vR}\{\overline{w'b'}/g - 0.61\overline{w'q'}\}$$

$$= (300\ \text{K})\{(3.5\text{x}10^{-2}\ \text{m}^2\text{s}^{-3})/(9.8\ \text{m s}^{-2}) - 0.61(10^{-4}\ \text{m s}^{-1})\}$$

$$= 8.9\text{x}10^{-2}\ \text{K m s}^{-1},$$

so $\delta\theta = 2\overline{w'\theta'}\ (h/2)/V = \underline{0.12}$ K. This variability is quite small. It is fairly realistic, though in reality w' is not perfectly correlated with q' and θ', so to support the required fluxes, the thermodynamic perturbations must be a little larger than the above argument suggests.

(f) The temporal sampling rate $f = 25$ Hz corresponds to a distances $l = u_{plane}/f = 4$ m between samples. Heuristically, we expect the intersample differences to be due mainly to eddies of scale l between updrafts and downdrafts. If v_l is the typical velocity scale of such eddies, and if they are in the inertial range of scales, the Kolmogorov cascade implies that the rate of energy flow to smaller scales is $\varepsilon = -D$ regardless of scale, so