Lecture 7: The Complex Fourier Transform and the Discrete Fourier Transform (DFT)

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7.1 Fourier analysis and filtering

Many data analysis problems involve characterizing data sampled on a regular grid of points, e. g. a *time series* sampled at some rate, a 2D image made of regularly spaced pixels, or a 3D velocity field from a numerical simulation of fluid turbulence on a regular grid.

Often, such problems involve characterizing, detecting, separating or manipulating variability on different scales, e. g. finding a weak systematic signal amidst noise in a time series, edge sharpening in an image, or quantifying the energy of motion across the different sizes of turbulent eddies.

Fourier analysis using the **Discrete Fourier Transform** (DFT) is a fundamental tool for such problems. It transforms the gridded data into a linear combination of oscillations of different wavelengths. This partitions it into scales which can be separately analyzed and manipulated.

The computational utility of Fourier methods rests on the **Fast Fourier Transform** (FFT) algorithm, developed in the 1960s by Cooley and Tukey, which allows efficient calculation of discrete Fourier coefficients of a periodic function sampled on a regular grid of 2^p points (or $2^p 3^q 5^r$ with slightly reduced efficiency).

7.2 Example of the FFT

Using Matlab, take the FFT of the HW1 wave height time series (length $24 \times 60 = 2^5 3^2 5$) and plot the result (Fig. 1):

load hw1_dat; zhat = fft(z); plot(abs(zhat),'x')

A few elements (the second and last, and to a lesser extent the third and the second from last) have magnitudes that stand above the noise. Something is there...but what? The next few lectures will be devoted to interpreting and using the DFT.

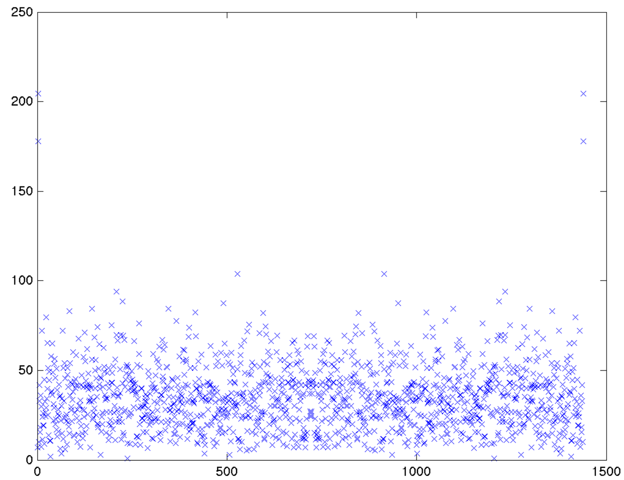

Figure 1: Absolute value of DFT of wave height time series of HW1

7.3 Complex Fourier series

A piecewise-continuous L-periodic function $u(t)$ has a convergent Fourier series

$$u(t) = a_0/2 + \sum_{M=1}^{\infty} \left[a_M \cos(2\pi M t/L) + b_M \sin(2\pi M t/L) \right],$$

where

$$(a_M, b_M) = \frac{2}{L} \int_0^L u(t) \left(\cos(2\pi M t/L), \sin(2\pi M t/)L \right) dt, \;\; M = 0, 1, 2, 3, ...$$

Define **complex Fourier components**

$$c_{\pm M} = (a_M \mp i b_M)/2 = \frac{1}{L} \int_0^L u(t) e^{\mp 2\pi i M t/L} \, dt$$

and the **angular frequency** of each mode

$$\omega_M = 2\pi M/L$$

Then

$$\begin{aligned} a_M \cos(2\pi M t/L) \;\; &+ \;\; b_M \sin(2\pi M t/L) \\ &= \frac{a_M}{2} \left(e^{i\omega_M t} + e^{-i\omega_M t} \right) + \frac{b_M}{2i} \left(e^{i\omega_M t} - e^{-i\omega_M t} \right) \\ &= c_M e^{i\omega_M t} + c_{-M} e^{-i\omega_M t}, \\ u(t) &= L^{-1} \sum_{M=\infty}^{\infty} c_M e^{i\omega_M t}, \end{aligned} \tag{7.3.1}$$

which is the **complex Fourier series** for $u(t)$.

Exactly the same methodology applies to a periodic function of a spatial coordinate x. In that case, the terminology is to say **wavenumber** $k_M = 2\pi M/L$ in place of angular frequency ω_M.

7.4 Discrete Fourier Transform (DFT) and FFT

Let $u_j, j = 1, \ldots, N$ be a sequence of N possibly complex values. The **Discrete Fourier Transform** (DFT) of this sequence is the sequence $\hat{u}_m, m = 1, \ldots, N$, where

$$\hat{u}_m = \sum_{j=1}^{N} u_j e^{-2\pi i(m-1)(j-1)/N} \tag{7.4.1}$$

The inverse discrete Fourier transform (IDFT) is

$$u_j = \frac{1}{N} \sum_{m=1}^{N} \hat{u}_m e^{2\pi i(m-1)(j-1)/N} \tag{7.4.2}$$

The FFT is a fast algorithm for computing the discrete Fourier transform for data lengths $N = 2^p$, taking $O(N \log_2 N)$ flops as compared with $O(N^2)$ flops for doing the computation directly using the above formulas. Versions of the FFT that are nearly as efficient also apply for $N = 2^p 3^q 5^r$.

To show that the IDFT really is the inverse of the DFT, we substitute eqn. (7.4.1) into (7.4.2), after changing the summation index in the former to J:

$$\begin{aligned} u_j &\stackrel{?}{=} \frac{1}{N} \sum_{m=1}^{N} e^{2\pi i(m-1)(j-1)/N} \sum_{J=1}^{N} u_J e^{-2\pi i(m-1)(J-1)/N} \\ &= \frac{1}{N} \sum_{m=1}^{N} \sum_{J=1}^{N} u_J e^{2\pi i(m-1)(j-J)/N} \\ &= \frac{1}{N} \sum_{J=1}^{N} u_J \sum_{m=1}^{N} e^{2\pi i(m-1)(j-J)/N} \end{aligned} \tag{7.4.3}$$

The inner sum over m is a geometric series with ratio $\exp[2\pi i(m-1)(j-J)/N]$. If $J = j$, each term is 1 so the series sums to N. If $J \neq j$, the sum is:

$$\begin{aligned} \sum_{m=1}^{N} e^{2\pi i(m-1)(j-J)/N} &= \frac{\exp\left[\frac{2\pi i}{N} N(j-J)\right] - 1}{\exp\left[\frac{2\pi i}{N}(j-J)\right] - 1} \\ &= 0 \ (J \neq j) \end{aligned}$$

Thus, the only term in the outer sum over J that contributes is from $J = j$, for which the inner sum is N, and we indeed find that the RHS of (7.4.3) is equal to u_j as claimed.

7.5 Relation of DFT to complex Fourier series

Define a regular grid with N points across the period L:

$$t_j = (j-1)\Delta t,\ j = 1, ..., N, \qquad \Delta t = L/N.$$

We now show that the elements of the DFT are Riemann sum discretization of the complex Fourier transform integrals on this grid. We associate indices m in the DFT with the N lowest positive and negative harmonics:

$$m = 1, 2, \ldots, N \ \leftrightarrow\ M_m = 0, 1, 2, \ldots, N/2-1, -N/2, \ldots, -1, \tag{7.5.1}$$

Then the Riemann sum for the complex Fourier coefficient c_M is

$$\begin{aligned} c_M &\approx \frac{1}{L}\sum_{j=1}^{N} \Delta t u(t_j) \exp\left[-i\omega_M t_j\right] \\ &= \frac{1}{L}\sum_{j=1}^{N} \frac{L}{N} u(t_j) \exp\left[-i\frac{2\pi M}{L}(j-1)\frac{L}{N}\right] \end{aligned} \tag{7.5.2}$$

This Riemann approximation will be accurate as long as the integrand $u(t)\exp(-i\omega_M t)$ only varies slightly in a grid spacing. This requires that $u(t)$ be a smooth function of t on the grid and that the exponential $\exp(i\omega_M t)$ also not vary too much between grid points, i. e. that $|\omega_M \Delta t|$ $(= 2\pi|M|/N) \ll 1$. The complex Fourier coefficients of a smooth L-periodic $u(t)$ will decrease rapidly with increasing $|M|$. If N has been chosen so that all the Fourier modes with significant amplitude have $|M| \ll N/2$, these Fourier coefficients will be well approximated with the DFT using the correspondence (7.5.1.)

If $0 \le M < N/2$, we can set $M = m-1$ in (7.5.2) and use the definition for the corresponding DFT coefficient:

$$\begin{aligned} c_M &\approx \frac{1}{N}\sum_{j=1}^{N} u(t_j) \exp\left[-\frac{2\pi i}{N}(m-1)(j-1)\right] \\ &= \hat{u}_m/N. \end{aligned} \tag{7.5.3}$$

For the negative harmonics $-N/2 \le M < 0$, following (7.5.1) we associate $M = m-1-N$ and

$$\begin{aligned} c_M &\approx \frac{1}{N}\sum_{j=1}^{N} u(t_j) \exp\left[-\frac{2\pi i}{N}(m-1-N)(j-1)\right] \\ &= \frac{1}{N}\sum_{j=1}^{N} u(t_j) \exp\left[-\frac{2\pi i}{N}(m-1)(j-1)\right] \underbrace{\exp\left[\frac{2\pi i}{N}N(j-1)\right]}_{1} \\ &= \hat{u}_m/N \text{ again.} \end{aligned}$$