Lecture 8: Properties of the DFT

Winter 2015

8.1 Aliasing

Because the DFT is based on sampling a continuous function at a finite set of equally-spaced points $j\Delta t$, many different L-periodic functions can have the same DFT. In fact, different sinusoids can have the same DFT, an ambiguity called *aliasing*. In general, consider harmonic M sampled on the grid $t_j = (j-1)\Delta t$ where $j = 1, ..., N$ and $\Delta t = L/N$:

$$\exp(i\omega_M t_j) = \exp\left[i\frac{2\pi M}{L}(j-1)\frac{L}{N}\right] = \exp\left[2\pi i\frac{M(j-1)}{N}\right]$$

Harmonics $M + qN, \quad q = \pm 1, \pm 2, ...$ also have exactly the same values at the grid points, since

$$\begin{aligned} \exp(i\omega_{M+qN}t_j) &= \exp\left[2\pi i\frac{(M+qN)(j-1)}{N}\right] \\ &= \exp\left[2\pi i\frac{M(j-1)}{N}\right]\cdot\exp\left[2\pi iq(j-1)\right] \\ &= \exp(i\omega_M t_j) \end{aligned} \qquad (8.1.1)$$

Thus there is always ambiguity in whether one is looking at a smooth, adequately sampled signal, or a highly oscillatory, poorly sampled signal. Our choice of how to assign harmonics M to the different components m of the DFT was based on assuming the signal is smooth. A signal composed of highly oscillatory harmonics that alias to a low harmonic M on the given grid will just add to the true signal from that harmonic. For instance DFT element $m = 1$ includes the signal not only from harmonic $M = 0$ but also from $M = \pm N, \pm 2N, ...,$ as shown in the left panel of Fig. 1.

The *Nyquist frequency* $\omega_{N/2} = \pi/\Delta t$ is the maximum frequency that is unambiguously detectable on the grid. It corresponds to a (1, -1, 1, -1...) oscillation on the grid of period $2\Delta t$ (right panel of Fig. 1).

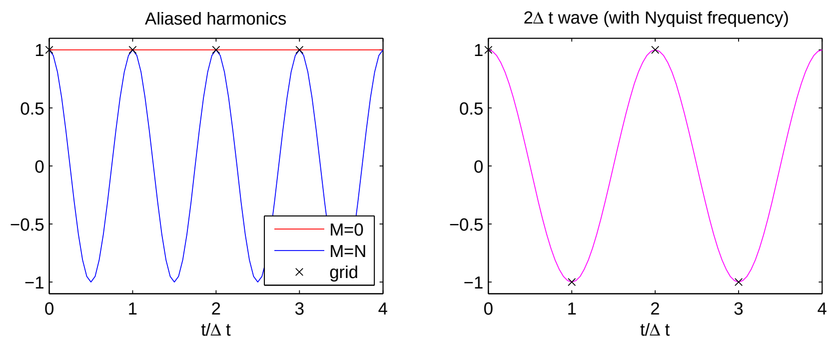

Figure 1: Left: Harmonic N, which is a sinusoid with period Δt, aliases to (has the same grid point values as) harmonic 0, which is a constant. Right: A sinusoid with the Nyquist frequency.

8.2 Matrix form of DFT/IDFT; Parseval's Thm

The DFT and IDFT can be expressed in matrix form. If $\mathbf{u}$ is the vectors of gridpoint values u_j, then:

$$\hat{\mathbf{u}} = DFT(\mathbf{u}) = N^{1/2} F\mathbf{u}, \tag{8.2.1}$$

$$\mathbf{u} = IDFT(\hat{\mathbf{u}}) = N^{-1/2} F^{\dagger}\hat{\mathbf{u}}, \tag{8.2.2}$$

where $\hat{\mathbf{u}}$ is the DFT of $\mathbf{u}$, the elements of the DFT matrix F are

$$F_{mj} = N^{-1/2} \exp(-2\pi i(m-1)(j-1)/N), \tag{8.2.3}$$

and $F^{\dagger}$ is the conjugate transpose of F.

We showed above that the IDFT is the inverse of the DFT, so

$$\mathbf{u} = N^{-1/2} F^{-1}\hat{\mathbf{u}} \ \Rightarrow \ F^{-1} = F^{\dagger}. \tag{8.2.4}$$

That is, F is a *unitary* matrix. This gives an easy derivation of *Parseval's theorem*

$$\begin{aligned} \sum_{m=1}^{N} |(\hat{u}_m/N)^2| &= \hat{\mathbf{u}}^{\dagger}\hat{\mathbf{u}}/N^2 \\ &= \mathbf{u}^{\dagger}F^{\dagger}F\mathbf{u}/N \\ &= \mathbf{u}^{\dagger}\mathbf{u}/N \\ &= N^{-1}\sum_{j=1}^{N}|u_j^2|. \end{aligned} \tag{8.2.5}$$

That is, the sum of the squares of the approximate Fourier coefficients $\hat{u}_m/N$ is equal to the average *power* or squared amplitude of the time series u_j. **We**

interpret Parseval's theorem as a partitioning of the power into contributions from each harmonic or wavenumber; this is very useful for interpretation of data.

8.3 Key things to remember about the DFT

Matlab DFT: **uhat = fft(u)**; inverse DFT: **u = ifft(uhat)**.
Will calculate the DFT or inverse DFT using the 'fast' algorithm if the data length is $N = 2^p 3^q 5^r$. For other N, it will take $O(N^2)$ flops and go much slower if N is large.

Assumes periodic input : $u_{N+1} = u_1$ (discontinuities between the endpoints can create unintended artifacts)

Relation to Fourier series If **u** is sampled from a continuous periodic function $u(t)$, **uhat**/N gives an estimate of its complex Fourier series coefficients c_M:

$$\hat{u}_m/N \approx c_M, M = m-1 \ (1 \le m \le N/2) \text{ or } m-1-N \ (N/2+1 \le m \le N).$$

For smooth functions $u(t)$ and low-order harmonics, this approximation is extremely accurate. Parseval's theorem partitions the power in **u** into the Fourier modes or harmonics in its DFT.

Account for the shift between the indices m and the corresponding Fourier harmonics M_m. In Matlab, define the index vector of harmonics **M = [0:(N/2-1) -N/2:-1]** and the frequencies **om = 2*pi*M/L** (or wave numbers **k = 2*pi*M/L** in a problem in which position x is the independent variable).

m=1 coefficient of uhat is N times the mean of **u** (easily proved from DFT definition).

DFT is complex-valued If **u** is real, the DFT coefficients for Fourier modes M and $-M$ are complex conjugates (easily proved from DFT definition).

x **derivative of spatially periodic function** Matlab: **dudx = real(ifft(1i*k*fft(u)))**;