Atm S 552 Lecture 14: Cross-spectral analysis

Chris Bretherton

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Reference: Hartmann Atm S 552 notes, Chapter 6.1-2.

1 The cross-spectrum

The cross-spectrum is a simple generalization of univariate spectral analysis to study frequency-dependent covariance and correlation between two time series. From N samples $\{u_j\}$ and $\{v_j\}$, $j = 1, ..., N$, we can construct the *cross-spectral periodogram*:

$$P_{uv}(f_m) = \hat{u}_m \hat{v}_m^* / (N^2 \Delta f), \qquad m = 1, ..., N \tag{1}$$

$P_{uv}(f_m)$ is complex-valued, but as long as both time series are real-valued, $\hat{u}(f_m) = \hat{u}^*(-f_m)$ and similarly for v, so

$$P_{uv}(f_m) = P_{uv}^*(-f_m). \tag{2}$$

Note that P_{uu} is the regular periodogram of $\{u_j\}$.

If the time series are samples from stationary, mutually correlated, random processes U and V, then as $N \to \infty$, the expected value of $P_{uv}(f)$ at any given frequency f converges to an underlying true cross-spectral density $P_{UV}(f)$.

The cross-spectral density has two important and related applications:

1. Decomposing the covariance between U and V into frequency components
2. Using U to explain a part of the variability of V based on their frequency-resolved **coherence**

$$Coh_{UV}(f) = \frac{P_{UV}(f)}{(P_{UU}(f) P_{VV}(f))^{1/2}} \tag{3}$$

2 Parseval Theorem for Covariance

Parseval's theorem can be generalized to partition the biased covariance of $\{u_j\}$ and $\{v_j\}$ into a sum of contributions from the different frequencies of the DFT:

$$\sum_{j=1}^{N} u_j v_j / N = \sum_{m=1}^{N} \hat{u}_m \hat{v}_m^* / N^2 = \sum_{m=1}^{N} P_{uv}(f_m) \Delta f \tag{4}$$

This is proved using the matrix form of the DFT, as in Lecture 8.2:

$$\begin{aligned}\sum_{m=1}^{N} \hat{u}_m \hat{v}_m^* / N^2 &= \hat{\mathbf{v}}^\dagger \hat{\mathbf{u}} / N^2 \\ &= \mathbf{v}^\dagger F^\dagger F \mathbf{u} / N \\ &= \mathbf{v}^\dagger \mathbf{u} / N \\ &= \sum_{j=1}^{N} u_j v_j / N. \end{aligned} \tag{5}$$

The $\mathbf{v}^\dagger$ is just a transpose since $\mathbf{v}$ is real-valued.

As with the univariate periodogram, the cross-spectral periodogram is a noisy estimator of the true cross-spectral density $P_{UV}(f)$ and requires further averaging, e. g. windowed Fourier analysis, other spectral estimation methods such as multitaper methods, or (for broad-band processes) octave averaging, to be quantitatively useful.

3 Cospectrum and quadrature spectrum

The cross-spectrum (true or single-sample) can be written

$$P_{UV}(f) = C_{UV}(f) + iQ_{UV}(f) \tag{6}$$

Here, $C_{UV}(f)$ is the **cospectrum** and $Q_{UV}(f)$ is the **quadrature spectrum**. A complex exponential of frequency f with any phase lag ϕ can be written as a sum of in-phase and quadrature parts:

$$e^{i(ft+\phi)} = e^{ift} \cos\phi + ie^{ift} \sin\phi$$

so their combination can describe any linear lagged relationship between U and V at a given frequency f. The cospectrum can be viewed as the simultaneous covariance between U and V at frequency f and the quadrature spectrum is the covariance between U and V lagged by a phase $\pi/2$ at frequency f.

4 Interpretation of the coherence as a frequency-dependent correlation

From this discussion, we can interpret $Coh_{UV}(f)$ as being a form of frequency-dependent correlation coefficient between U and V, but also including through its imaginary part any out-of-phase lagged correlation between them. The **magnitude-squared coherence**

$$|Coh_{UV}|^2(f) = C_{UV}^2 + Q_{UV}^2 \tag{7}$$

can be thought of as the fraction of variance of V that can be explained by linear, possibly phase-lagged frequency-dependent covariability of U:

$$\hat{V}(f) = \hat{a}(f)\hat{U}(f) + \hat{N}(f) \tag{8}$$

where $\hat{a}(f)$ is the *transfer function* and $\hat{N}(f)$ is noise uncorrelated with U. Multiplying by $\hat{U}_m^*$ and taking the expectation, we see that the transfer function is related to the cross-spectrum,

$$\hat{a}(f) = \frac{P_{UV}(f)}{P_{UU}(f)} \tag{9}$$

The fraction of the variance of $\hat{V}(f)$ explained by this linear relationship is

$$\begin{aligned} EVF(f) &= \frac{|\hat{a}|^2 E[|\hat{U}|^2]}{E[|\hat{V}|^2]} = \frac{|P_{UV}|^2}{P_{UU}^2}\frac{P_{UU}}{P_{VV}} \\ &= |Coh|^2(f) \end{aligned} \tag{10}$$

5 Example

The script **CrossSpectralNino3NPI.m** applies cross-spectral analysis to compare deseasonalized monthly anomalies of Nino3.4 (central equatorial Pacific) SST with the North Pacific Index (NPI), defined as the mean sea-level pressure over the region 30°N-65°N, 160°E-140°W. The NPI is strongly correlated to Pacific Northwest wintertime weather, with positive NPI corresponding to cooler, more northwesterly flow and more mountain snowfall. The NPI has much more monthly variability than Nino3.4, reflecting strong internal atmospheric variability. However, there is systematically negative cospectral power (anticorrelation of Nino3.4 and NPI) at periods of 2 years and longer, such that El Nino (positive Nino3.4 SSTA) favors a negative NPI (a warmer, less snowy Pacific Northwest).