Lecture 18: Wavelet Analysis

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Refs: Matlab Wavelet Toolbox help.

18.1 Introduction

Wavelets are an efficient tool for analyzing data that varies on a wide range of scales, especially when the data is statistically non-stationary, e. g. speech, pictures, etc. In such situations they are computationally more efficient and easy to use than windowed Fourier methods. Wavelets are also an attractive and widely-used way to compress such data.

Idea: A form of **multi-resolution analysis**. Decompose data into a sum of time series which characterize the variability on different time scales, each twice as long as the previous one, using an algorithm called a **discrete wavelet transform (DWT)**. The idea is to filter the time series by multiplying it by a localized function called a *wavelet* whose width in time can be rescaled to pick out variability on the different time scales. Haar (1909) coined the word 'wavelet', but the approach was popularized by Morlet and Daubechies in the 1980s, furthered by development of a fast DWT/inverse DWT algorithm (Mallat 1988) and the arrival of big data whose computational analysis benefitted from this approach. *Multigrid methods* for numerically solving elliptic PDEs have a similar philosophical foundation.

An analysis technique called the Continuous Wavelet Transform (CWT; Matlab Wavelet Toolbox function **cwt**) is popular for visualizing (rather than quantifying) time-frequency behavior. Generally, I prefer the DWT as a more parsimonious description of this behavior.

18.2 The Haar transform

We consider a time series $\mathbf{u} = u_1, \ldots, u_N$, where N is even. The single-level Haar transform decomposes $\mathbf{u}$ into two signals of length $N/2$. They are the **average** coefficient vector $\mathbf{a}^1$, with components

$$a_m = 2^{-1/2}(u_{2m-1} + u_{2m}), \qquad m = 1, \ldots, N/2, \tag{18.2.1}$$

and the **detail** coefficient vector $\mathbf{d}^1$, with components

$$d_m = 2^{-1/2}(u_{2m-1} - u_{2m}), \qquad m = 1, \ldots, N/2. \tag{18.2.2}$$

Each term in the detail vector represents variations between successive elements of the time series, i. e. on a time scale Δt. Each term in the average vector is an average across a time scale $2\Delta t$. These can be concatenated into another N-vector, which can be regarded as a linear matrix transformation of $\mathbf{u}$:

$$\mathbf{h}^1 = [\mathbf{a}^1|\mathbf{d}^1] \tag{18.2.3}$$

This transform can be inverted to get $\mathbf{u}$ from $\mathbf{h}$, as follows:

$$u_{2j-1} = 2^{-1/2}(a_j + d_j), \;\; u_{2j} = 2^{-1/2}(a_j - d_j), \;\; j = 1, \ldots, N/2. \tag{18.2.4}$$

Like the Fourier transform, the Haar transform is power-conserving (2-norm conserving):

$$\begin{aligned}
|\mathbf{h}^1|^2 &= \sum_{m=1}^{N/2} \left[a_m^2 + d_m^2\right] \\
&= \sum_{m=1}^{N/2} \frac{1}{2}\left[\{u_{2m-1} + u_{2m}\}^2 + \{u_{2m-1} - u_{2m}\}^2\right] \\
&= \sum_{m=1}^{N/2} \left[u_{2m-1}^2 + u_{2m}^2\right] \\
&= |\mathbf{u}|^2
\end{aligned} \tag{18.2.5}$$

This means that the Haar transform can be regarded as partitioning the power between different time scales and time ranges.

18.3 Haar wavelets and reconstruction of the signal from wavelet and scaling coefficients

It is useful to develop some more formalism for discussing the Haar transform. The detail components can be written as an inner product of the time series with the m'th **level-1 Haar wavelet** $\mathbf{W}_m^1$,

$$d_m = \mathbf{W}_m^1 \cdot \mathbf{u}, \tag{18.3.1}$$

where:

$$\begin{aligned}
\mathbf{W}_1^1 &= 2^{-1/2}[1, -1, 0, 0, 0, \ldots, 0, 0] \\
\mathbf{W}_2^1 &= 2^{-1/2}[0, 0, 1, -1, 0, \ldots, 0, 0] \\
&\vdots \\
\mathbf{W}_{N/2}^1 &= 2^{-1/2}[0, 0, 0, 0, 0, \ldots, 1, -1]
\end{aligned} \tag{18.3.2}$$

Each level-1 Haar wavelet is:

1. A translation of $\mathbf{W}_1^1$ by an even number of units.
2. Orthogonal to all other Haar wavelets

Similarly, the average components can be written

$$a_m = \mathbf{V}_m^1 \cdot \mathbf{u}, \tag{18.3.3}$$

where t

$$\begin{aligned} \mathbf{V}_1^1 &= 2^{-1/2}[1, 1, 0, 0, 0, \ldots, 0, 0] \\ \mathbf{V}_2^1 &= 2^{-1/2}[0, 0, 1, 1, 0, \ldots, 0, 0] \\ &\vdots \\ \mathbf{V}_{N/2}^1 &= 2^{-1/2}[0, 0, 0, 0, 0, \ldots, 1, 1] \end{aligned} \tag{18.3.4}$$

are the level-1 **Haar scaling signals**, which are also translations of each other that are mutually orthogonal and orthogonal to all the level-1 wavelets.

Using this notation, we can write the signal in terms of components proportional to the wavelets and scaling signals:

$$\mathbf{u} = \mathbf{A}^1 + \mathbf{D}^1 \tag{18.3.5}$$

where

$$\mathbf{A}^1 = 2^{-1/2}(a_1, a_1, a_2, a_2, \ldots, a_{N/2}, a_{N/2}) = \sum_{m=1}^{N/2} a_m \mathbf{V}_m^1 \tag{18.3.6}$$

and

$$\mathbf{D}^1 = 2^{-1/2}(d_1, -d_1, d_2, -d_2, \ldots, d_{N/2}, -d_{N/2}) = \sum_{m=1}^{N/2} d_m \mathbf{W}_m^1 \tag{18.3.7}$$

This vectorially describes how to invert the Haar transform to get the original time series.

18.4 Single-level DWT in Matlab

The Matlab wavelet toolbox has an extensive set of functions for wavelet analysis. The Matlab script **wavelet_turbulence** (class web page) uses these on a nonstationary time series of aircraft-measured vertical velocity. The first part of this script does a single-level Haar wavelet analysis of this time series.

18.5 Multiresolution analysis with Haar transform

By applying the Haar transform to the average coefficient vector, we could decompose that into a level-2 average vector (aggregation to $2^2\Delta t$) and a level-2 detail vector (variation on scale $2^{2-1}\Delta t$), each of length $N/4$. If $N = 2^P$ one can continue this process up to level P, at which there is one average (representing aggregation to scale $2^P\Delta t$) and one detail coefficient (representing variation on the scale $2^{P-1}\Delta t$). This is called **multiresolution analysis**.

18.6 Higher-level wavelets and scaling vectors

For a level-2 analysis, we apply the Haar transform to the level-1 average vector $\mathbf{a}^1$ of length $N/2$:

$$\begin{aligned} d^2_m &= 2^{-1/2}(a_{2m-1} - a_{2m}) = 2^{-1}(u_{4m-3} + u_{4m-2} - u_{4m-1} - u_{4m}) \\ a^2_m &= 2^{-1/2}(a_{2m-1} + a_{2m}) = 2^{-1}(u_{4m-3} + u_{4m-2} + u_{4m-1} + u_{4m}) \end{aligned} \quad (18.6.1)$$

We can rewrite this:

$$\begin{aligned} d^2_m &= W^2_m \cdot \mathbf{u} \\ a^2_m &= V^2_m \cdot \mathbf{u} \end{aligned} \quad (18.6.2)$$

where the **level-2 wavelets** are:

$$\begin{aligned} \mathbf{W}^2_1 &= 2^{-1}[1, 1, -1, -1, 0, 0, 0, 0, 0, \ldots, 0, 0] \\ \mathbf{W}^2_2 &= 2^{-1}[0, 0, 0, 0, 1, 1, -1, -1, 0, \ldots, 0, 0] \\ &\vdots \\ \mathbf{W}^2_{N/4} &= 2^{-1}[0, 0, 0, 0, 0, \ldots, 1, 1, -1, -1] \end{aligned} \quad (18.6.3)$$

and the **level-2 scaling signals** are :

$$\begin{aligned} \mathbf{V}^2_1 &= 2^{-1}[1, 1, 1, 1, 0, 0, 0, 0, 0, \ldots, 0, 0] \\ \mathbf{V}^2_2 &= 2^{-1}[0, 0, 0, 0, 1, 1, 1, 1, 0, \ldots, 0, 0] \\ &\vdots \\ \mathbf{V}^2_{N/4} &= 2^{-1}[0, 0, 0, 0, 0, \ldots, 1, 1, 1, 1] \end{aligned} \quad (18.6.4)$$

Again, these vectors are all mutually orthogonal and also orthogonal to the level-1 wavelets.

The inverse of this 2-level Haar transform can be expressed

$$\mathbf{u} = \underbrace{\mathbf{A}^2 + \mathbf{D}^2}_{\mathbf{A_1}} + \mathbf{D}^1. \quad (18.6.5)$$

where

$$\mathbf{A}^2 = 2^{-1}(a^2_1, a^2_1, a^2_1, a^2_1, \ldots, a^2_{N/4}, a^2_{N/4}, a^2_{N/4}, a^2_{N/4}) = \sum_{m=1}^{N/2} a^2_m \mathbf{V}^2_m \quad (18.6.6)$$

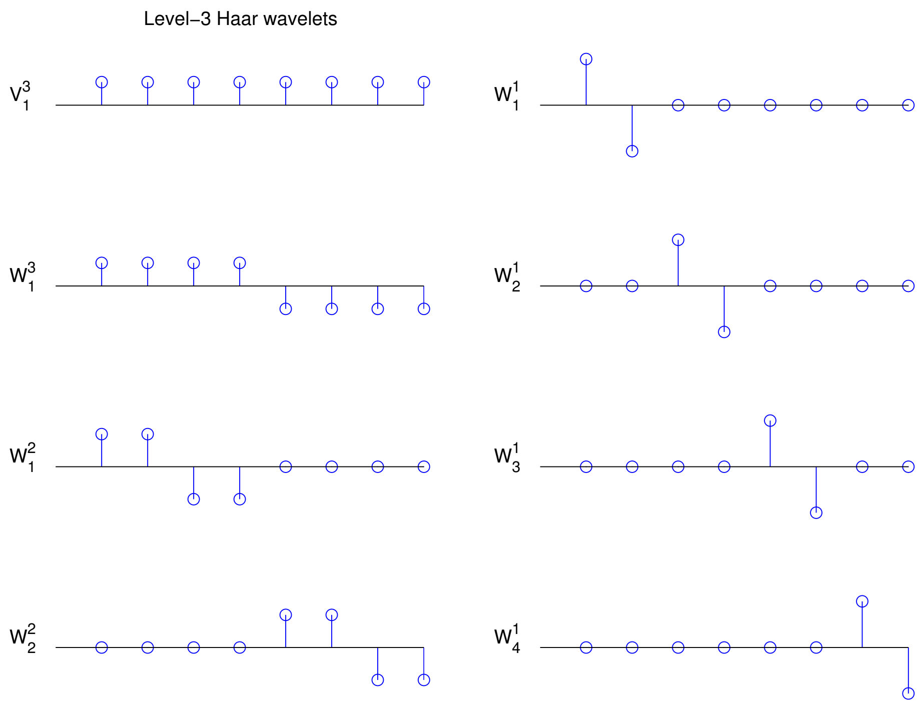

Figure 1: The wavelets and the scaling vector for a level-3 analysis of a time series of length 8.

and similarly

$$\mathbf{D}^2 = \sum_{m=1}^{N/2} d_m^2 \mathbf{W}_m^2 \tag{18.6.7}$$

The extension to P levels should be clear. The P-level Haar transform of the data vector is

$$\mathbf{h}^P = [\underbrace{\mathbf{a}^P}_{N/2^P} \,|\, \underbrace{\mathbf{d}^P}_{N/2^P} \,|\, \underbrace{\mathbf{d}^{P-1}}_{N/2^{P-1}} \,| \cdots |\, \underbrace{\mathbf{d}^1}_{N/2}] \tag{18.6.8}$$

and Fig. 1 shows the Haar wavelet and scaling vectors for a $P = 3$ multiresolution analysis.

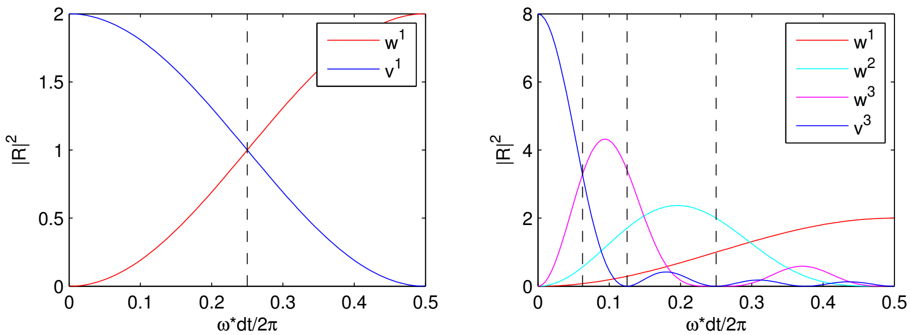

Figure 2: (a) Spectral power of equivalent filters for the details and averages in a 1-level Haar transform; (b) Same, but for a 3-level Haar transform. Black dashed lines denote periods $4\Delta t, 8\Delta t, 16\Delta t$.

18.7 Interpretation of Haar wavelets in terms of filters

We can interpret the level-1 Haar detail and average vectors as filtering with the weight vectors

$$\begin{aligned} \mathbf{w}^1 &= [w_0, w_1, \ldots, w_{-1}] = 2^{-1/2}[1, -1, 0, 0, \ldots, 0] \\ \mathbf{v}^1 &= [v_0, v_1, \ldots, v_{-1}] = 2^{-1/2}[1, 1, 0, 0, \ldots, 0] \end{aligned} \quad (18.7.1)$$

and then *binary subsampling* by keeping only every second component of the filtered fields. The spectral responses or these filters are simply obtained from the Z-transform

$$\begin{aligned} R_w(f) &= 2^{-1/2}(1 - Z)|_{Z=\exp(-2\pi i f \Delta t)} = 2^{1/2} i e^{-i\pi f \Delta t} \sin(\pi f \Delta t) \\ R_v(f) &= 2^{-1/2}(1 + Z) = 2^{1/2} e^{-i\pi f \Delta t} \cos(\pi f \Delta t) \end{aligned} \quad (18.7.2)$$

The spectral power of each of these filters is seen in Figure 1a. The level-1 wavelet filter is a coarse high-pass filter retaining frequencies $1/4 < f\Delta t < 1/2$; the level-1 scaling filter is a coarse low-pass filter retaining frequencies $f\Delta t < 1/4$.

The levels $p = 1, , P$ of the multilevel Haar transform can be also regarded as filtering followed by binary subsampling. Level-p wavelets behave as band-pass filters retaining frequencies $2^{-(p+1)} < f\Delta t < 2^{-p}$, while the scaling vectors behave as low-pass filters retaining frequencies $f\Delta t < 2^{-(p+1)}$. Fig. 1b shows the spectral power of the equivalent filters for each level for the case $P = 3$. The filter responses are consistent with the level-p wavelets extracting the fluctuations with periods between $2^p\Delta t$ and $2^{p+1}\Delta t$ in the time series.

18.8 Multiresolution wavelet analysis in Matlab

The second part of **wavelet_turbulence** shows a level-3 Haar wavelet analysis.

18.9 Continuous Wavelet Transform (CWT)

Given a continuous signal $u(t)$ and an **analyzing wavelet** $\psi(x)$, the CWT has the form

$$W(\lambda, t) = \lambda^{-1/2} \int_{-\infty}^{\infty} \psi(\frac{s-t}{\lambda}) u(s) ds \qquad (18.9.1)$$

Here λ, the scale, is a continuous variable. We insist that ψ have mean zero and that its square integrates to 1. The continuous Haar wavelet is defined:

$$\psi(t) = \begin{cases} 1 & 0 < t < 1/2 \\ -1 & 1/2 < t < 1 \\ 0 & \text{otherwise} \end{cases} \qquad (18.9.2)$$

$W(\lambda, t)$ is proportional to the difference of running means of u over successive intervals of length $\lambda/2$.

In practice, for a discrete time series, the integral is evaluated as a Riemann sum using the Matlab wavelet toolbox function **cwt**. The last section of **wavelet_turbulence** gives an example.