GFD1 AS509/OC512 Winter Q. 2017 Dargan Frierson
lab 2 P.B.Rhines Tues 17 Jan 12.30
Rotating fluids-I

In classical fluid dynamics, without the effects of rotation or density stratification, the fluid has no vorticity unless viscous forces or external forces produce it. If a fluid starts from rest (no velocity, no vorticity), it cannot develop vorticity. Viscous forces arise in boundary layers at rigid boundaries, and are often the main source of vorticity, and thus are important to the production of fluid dynamical drag and propulsion.

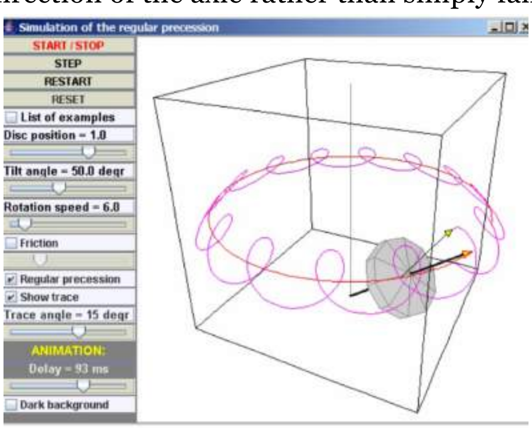
All this changes in a rotating fluid. Rotation of the fluid as a whole provides a background vorticity equal to twice the rotation rate $\vec{\Omega}$, regardless of viscosity. This omnipresent vorticity can be amplified and expressed in ocean eddies and atmospheric weather systems. The Earth's rotation vector $\vec{\Omega}$ has a definite polar axis, deciding the orientation of the planetary vortex lines.

It helps to consider a mechanical form of solid body rotation: a gyroscope. We use bicycle wheels to gain a sense of the dynamics of steady rotation. A steadily rotating point mass is accelerating, despite its constant speed: the direction of the velocity changes, requiring a force toward the center of rotation. Rotational motion involves angular momentum, call it **a.m.**, formed by taking the cross product of a position vector $\vec{R}$ from the center of rotation to the mass, and the velocity $\vec{U}$ which is perpendicular to $\vec{R}$. For the gyroscope these are perpendicular, so that *a.m.* is simply *MRU*, where U is the magnitude of $\vec{U}$. With the angular velocity of the wheel being $\vec{\Omega}$ the velocity of the mass is $\vec{U} = \frac{d\vec{R}}{dt} = \vec{\Omega} \times \vec{R}$. For a point mass, Newton's 2d law $\vec{F} = M\frac{d\vec{U}}{dt}$ becomes

$$\begin{aligned}\vec{R}\times\vec{F} &= \frac{d}{dt}(M\vec{R}\times\vec{U}) = \frac{d}{dt}(M\vec{R}\times(\vec{\Omega}\times\vec{R})) \\ &= \frac{d}{dt}(MR^2\vec{\Omega}) = \frac{d}{dt}(\mathrm{I}\,\vec{\Omega}) = \frac{d\vec{L}}{dt}\end{aligned} \quad . \qquad (*)$$

The angular momentum $\vec{L} = MR^2\vec{\Omega} = M\vec{R}\times\vec{U}$, which here is MRU, changes at a rate equal to the applied torque. M is the mass, $\vec{F}$ is an external force, $\vec{R}\times\vec{F}$ the external torque, and I = MR^2 is called the moment of inertia. Torque vectors perpendicular to $\vec{\Omega}$, such as the gravity force acting through the center of mass, or a twisting applied to the axle of the gyroscope, will change the direction but not the magnitude of the a.m. With the mass distributed about the rim of the wheel at radius R, the equations are the same. Changing the a.m. , by tilting the bicycle wheel, requires a torque which may need to be quite strong, particularly with heavy (water-filled) tires. More generally, for a rigid body of arbitrary shape, the moment of inertia I is a tensor

Standing on a lazy-susan platform, and twisting the rotating wheel, the change in angular momentum of the wheel is communicated to the person holding it. Strangest of all, the gyroscope set on the floor (or with its axle in the palm of your hand) does not fall down…it gradually rotates ('precesses') about the vertical. The torque of the gravity force on the tilted wheel is a horizontal vector at a right angle to the axle of the wheel. The gyroscope chooses the most efficient way to produce a.m. in the direction of the torque, by changing the direction of the axle rather than simply falling down.



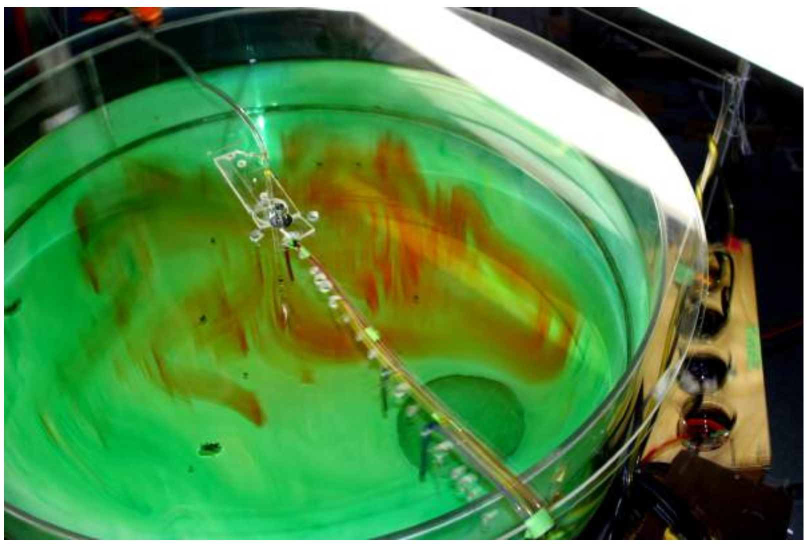
Numerical model of a gyroscope precessing about the vertical (Eugene Butikov) with and without nutation (looping oscillations)

The a.m. vector precesses to satisfy the equation of motion (*). While this is still not very intuitive, it helps to realize that if we start by holding the axle of the wheel fixed at an angle to the vertical, and then set it down on the floor, it *will* 'fall down' a little as it begins to precess. It will overshoot and oscillate gently ('nutation' or 'nodding') as it precesses, as in the figure above. The rate of slow precession of a tilted gyroscope is a vertical vector, say $\vec{\Omega}_P$, which is proportional to the gravity torque. Steady precession is given by

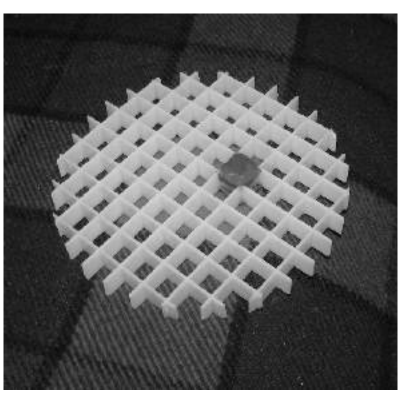
$$\frac{d\vec{L}}{dt} = \vec{\Omega}_P \times \vec{L} = \vec{R} \times \vec{F}$$

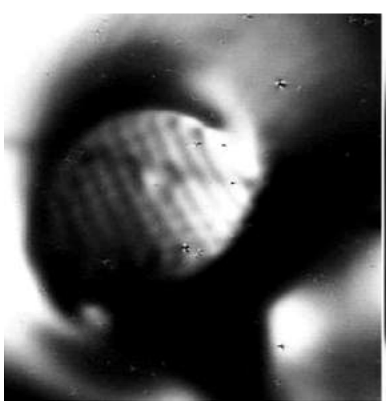
To get the direction right use the righthand screw rule for the cross product.

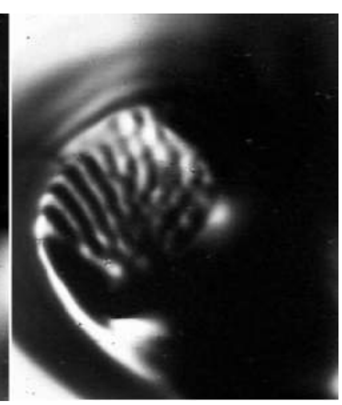
Taylor-Proudman flow. Each small vertical column of water on the rotating table has a.m. For fluid flows with small enough velocity, it requires a lot of torque to tilt over those columns, and hence when we stir the fluid it ends up moving in columns: 2-dimensional fluid flow. Colored dye in the image below lines up in ribbons that are thin when viewed from above. Think of the rotation as endowing the fluid with *stiffness*, which makes a vertical fluid column resist either deforming or tipping over.



The columns of fluid are so strongly constrained by their rotational stiffness that when they flow over a bump, they try to lift the water surface rather than reduce their height: the 'waffle grid' mountain below, left, sits on the bottom with 15 cm of water overhead. A *very slow* flow deforms the water surface in response, producing a distorted image of that shape (central two figures). This is imaged with a technique known as optical altimetry, which reveals the very small (hills and valleys ~ 1 micron = 10^{-6} m high) vertical distortion of the water surface. It is sort of a Salvador Dali image of the 'waffle'. The slow flow does not in fact manage to preserve the column heights, yet it leaves an imprint of the waffle shape in the surface above.







actual waffle

Similarly, if the flow encounters a taller mountain, it will try to go around rather than over it, in order to avoid changing the height of the water column; the fluid above the mountain acts like a tall, solid cylinder in deflecting the flow at all vertical levels. This 'virtual' cylinder of fluid is called a Taylor column. It is an extreme case for very slow flow.

Concentration of 'spin' (angular momentum). Beyond the strange and powerful nature of angular momentum of a spinning object, and its resistance to change, if we forcibly change the moment of inertia of a spinning mass, its rate of spin Ω will change to conserve a.m., $MRU \equiv MR^2\Omega$. This we experienced with water jugs held at arms length while rotating on the lazy-susan platform. Drawing them inward caused extreme acceleration.... the *figure-skater effect*. It is reversible, slowing the rotation when the arms are extended again.

Translated to the fluid, this same sequence of events occurs whenever fluid columns change their vertical length. Generally speaking, this leads to the *concentration* of the Earth's spin by the fluid, producing rotating weather systems, ocean eddies and jet-like concentrated flows...jet streams and oceanic boundary currents.

An example occurs with slightly faster flow encountering a change in fluid depth. Then, flow does succeed in rising to cross above the mountain in the figure below. . Our gyroscope experience shows that the vortex tube will try to remain vertical as it squeezes over the mountain top. In that case the height h of the fluid column must decrease, and its horizontal area A must increase to keep the volume of the vertical tube,

$$\text{volume} = A\,h = \text{constant}.$$

The spin of the column must decrease, while the fluid columns initially sitting above the mountain are stretched vertically as they flow into deeper water. Their spin increases.

Planetary and relative vorticity. The total vorticity of the fluid is the vector sum of the planetary and relative vorticities

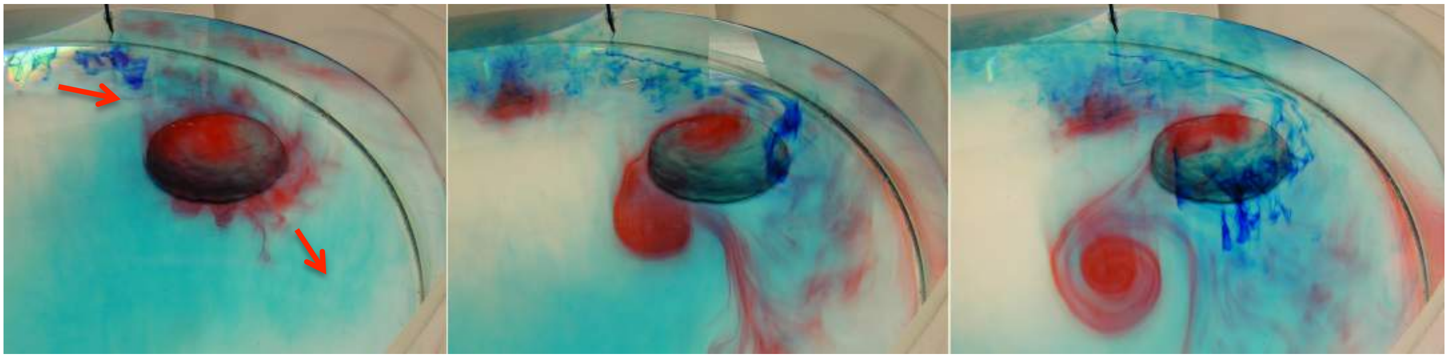
$$2\vec{\Omega} + \vec{\omega}; \quad \vec{\omega} \equiv \nabla \times \vec{u}$$

Angular momentum of a fluid column is proportional to its vertical vorticity only as long as the column remains circular. Fluid columns generally deform, after which their a.m. can be changed by pressure forces, so we must abandon that idea and consider the vorticity itself. The *vortex-tube strength* is the product of total vorticity and the horizontal area A of the tube, which with Ah being constant, is proportional to

$$q \equiv |2\vec{\Omega} + \vec{\omega}|/h \ .$$

This is also called *potential vorticity,* a direct generalization of the vortex-tube strength conservation in a classical, non-rotating fluid. Quite remarkably, the potential q governs a vast range of atmosphere/ocean flows, while remaining conserved following fluid particles until altered by viscous forces. It generalizes immediately and simply to fluids with density stratification.

For the flow over the mountain, vortex tubes are squashed and conservation of vortex-tube strength says that the relative vertical vorticity ω decreases to compensate for the decrease in h (Ω here is the background rotation of the fluid which does not change). Starting from rest, the block of fluid initially above the mountain moves downstream into deeper water and is stretched vertically. So, conversely, its relative vorticity increases. If initially the relative vorticity vanishes or is small this means that $\vec{\omega}$ is anticyclonic over the mountain and



A clockwise large scale flow (red arrows) encountering a mountain sheds a cyclonic vortex, while anticyclonic vorticity twists the flow above the mountain . This is a source of 'lee cyclo-genesis' in the atmosphere and ocean. GFD lab, UW.

cyclonic in the block of fluid in the lee the mountain, to conserve $(2\vec{\Omega}+\vec{\omega})$/h. The red cyclone in the figures originated in fluid above the mountain.

Coriolis force. How does this relate to the Coriolis force? The usual derivation uses a coordinate transformation to express the momentum equation for an observer rotating with the Earth. Consider a rotating mass, however, rotating nearly steadily but with a small extra azimuthal velocity, v'. The total azimuthal (round-and-round) velocity is

$$v = \Omega r + v'$$

where r is the radial distance. The acceleration toward the center of rotation is

$$v^2/r = \Omega^2 r + 2\Omega v' + v'^2/r$$

The first term is the mean centripetal acceleration, the second is the Coriolis acceleration $2\Omega v'$ (linear in v') and the third is the contribution to the centripetal acceleration (nonlinear in v'). The mean centripetal acceleration is one component of the gradient of the 'geopotential' function, $\Phi = ½\Omega^2 r^2$, a potential energy term that defines the 'horizontal' direction on a rotating planet. The Coriolis 'force; is an acceleration when viewed by a non-rotating observer. It arises naturally from angular momentum conservation.

The ratio of the nonlinear acceleration v'^2/r to the Coriolis acceleration is $v'/2\Omega r$. If we identify r with the horizontal length scale L of a fluid circulation, this becomes the *Rossby number*

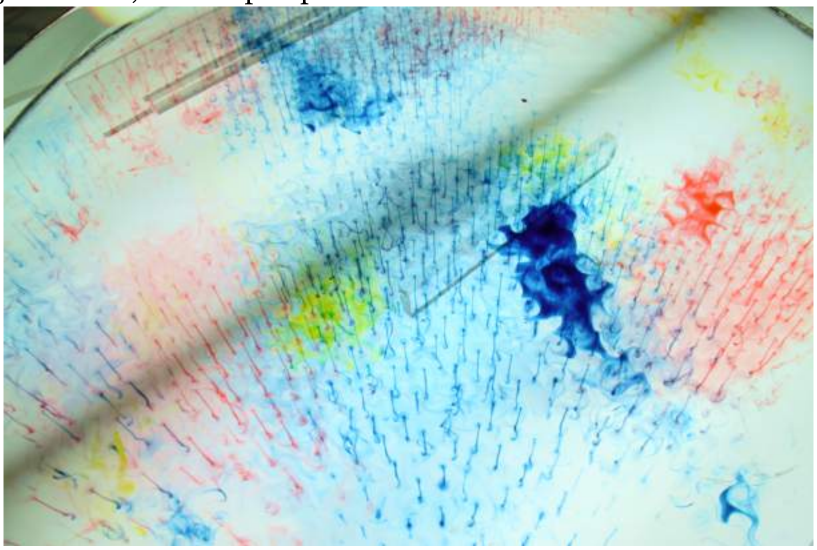
$$Ro = U/2\Omega L.$$

Natural flows with large scale, small velocity on a rapidly rotating planet have small Rossby number, Ro <<1. Putting in some numbers, a 10 m sec^{-1} wind in a weather system with L = 1000 km has a Rossby number $10/(10^{-4} \times 10^6) = 0.1$. Here we have fudged a bit, taking the local vertical component of $\vec{\Omega}$ at mid-latitude. An ocean circulation gyre with U = 0.1 m sec^1 and L =1000 km has a Rossby number even smaller, ~ 10^{-3}.

Fluid columns with small Rossby number become resistant to vertical stretching, yet if imposed, the stretching will have a strong effect. The mountain height in the experiment here is less than ¼ of the fluid depth. Yet it strongly affects the flow. This can be shown by estimating the fractional change in velocity, say dU/U caused by squeezing or stretching by an fractional amount dh/h. The answer, dU/U ~ Ro^{-1} (dh/h) which is much greater than dh/h, is left as an exercise for the reader.

In both examples, an observer who is observing the Earth but *not rotating with it* would say that the circulations have the same sense of rotation as the Earth (that is, the Earth's planetary vorticity >> the relative vorticity of the circulation. In essence, large-scale flows inherit the spin of the planet. A rotating observer does see both cyclonic and anticyclonic weather systems and ocean gyres but both appear cyclonic to the non-rotating observer. It is noteworthy that tropical cyclones and even tornadoes with diameter just a few hundred m. almost all rotate in the same sense as the Earth…cyclonically.

Vortex production by convection. It is useful to consider where the energy comes from, when studying a fluid circulation. We noticed that as the fluid in the rotating cylinder is cooled by surface evaporation, tiny tornadoes appear almost everywhere. The cooling only amounts to 0.1 °C or less, but it is enough to cause surface water to sink. This draws in fluid horizontally, which does the figure-skater thing, concentrating the planetary spin into a cyclonic vortex. Colored dye at the surface is drawn down the core of these vortices in the figure below, seen in perspective from above.



Convective vortices driven by surface cooling in a rapidly rotating fluid, where the sinking region forms a 'tree-trunk'

We also made vortices by removing fluid from the surface with a syringe or 'turkey-baster', thus stretching the vertical vortex lines. And, by floating snowballs at the surface. Their melting both cooled the surrounding water and made very dense melt water, leading to cyclonic vorticity which spun the snowballs anti-clockwise.

Energy increases when angular momentum is concentrated. Why are the 'tornadoes' so small? When the figure skater (with water jugs) draws their arms inward, spinning faster, angular momentum is conserved but the kinetic energy, KE, of the spin increases. This is supplied by the force drawing the weights inward: force x distance = change in mechanical energy. With such small buoyancy forcing, the convection cannot have much KE and thus cannot draw fluid in from a large radius. Hence, nature chooses small cells. The vorticity description of rotating convection is actually complex. Here the converging sinking at the surface stretches vertical vortex lines, making cyclonic relative vorticity. However at the bottom, the fluid diverges, flowing outward away from the vortex core. This makes some anticyclonic vorticity deep down, but boundary layer friction tends to reduce this.

Recall the discussion of dissipation of energy in the turbulent fluid of a stirred 'bath-tub': vortex tubes are stretched, likely increasing their length and $|\vec{\omega}|$ by factors of a thousand or more in a few seconds. Their cross-sectional area decreases accordingly, and the small-scale motions thus created are vulnerable to viscous forces, that can destroy the mechanical energy at a net rate $\iiint \nu |\vec{\omega}|^2 \, dV$; ν is the viscosity coefficient and dV a volume element. But before this happens, those small-scale eddies have gained much energy by the figure-skater mechanism, stealing it from the large eddies of the initial flow.