

2. Review of Linear Equatorial Dynamics
 Ref Matsuno, 1967b JMSS
 Lindzen 1967 MWR

A. Matsuno's barotropic free surface equation +
 the dispersion relationship

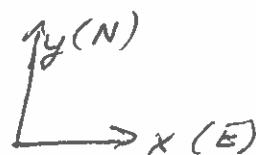
Matsuno (1966) considered the barotropic free surface equations, but noted the equivalence of this system to a ^{multi-}layer system or continuously stratified fluid:

$$u_t - f v + g \partial h / \partial x = 0 \quad (2.1)$$

$$v_t + f u + g h_y = 0 \quad (2.2)$$

$$h_t + H(u_x + v_y) = 0 \quad (2.3)$$

where (2.1, 2.2) are the ^{linearized} momentum equations for an incompressible fluid and h is a small perturbation from the mean depth H . ^{Subscripts are derivatives} A standard Cartesian coordinate system, ~~is used~~ centered on the equator is used.



If we define the geopotential $\phi = g h$ (2.4)
 and ^{assume} ~~write~~ β -plane geometry

$$f = \beta y; \quad \beta = 2.3 \times 10^{-11} \text{ s}^{-1} \text{ m}^{-1} \quad (2.5)$$

we can ~~and~~ nondimensionalize this set of equations

by ~~with~~ the length ^L and time ^T scales

$$T = (g/\beta)^{-1/2} \quad L = (c/\beta)^{1/2}$$

~~to the following eqns~~ Defining the nondimensional variables (')

$$u = cu', \quad v = cv', \quad \phi = c^2 \phi' \quad (x, y) = (c/\beta)^{1/2} (x', y')$$

$$t = (c\beta)^{-1/2} t' ; \quad c^2 = gH$$

eqns (2.1) - ~~through~~ (2.3) become, dropping the (') notation:

$$u_t - yv + \phi_x = 0 \quad (2.6)$$

$$v_t + yu + \phi_y = 0 \quad (2.7)$$

$$\phi_t + ux + vy = 0 \quad (2.8)$$

~~Eqs. for u and v~~ To obtain a single field eqn. we eliminate ϕ from (2.6) and (2.8), and (2.7) and (2.8)

$$v_{tt} - v_{yy} = -yu_t + u_{yx} \quad (2.9)$$

$$u_{tt} - u_{xx} = yv_t + \underbrace{v_{xy}}_{(2.9)} \quad (2.10)$$

Finally u is eliminated $\left[(\partial_{tt}^2 - \partial_{xx}^2)(2.9) \text{ and } (2.10) \right]$ to give the field eqn. [after integrating once in time]...

$$v_{yyt} - v_{ttt} + v_{xxt} + v_x - y^2 v_t = 0 \quad (2.11)$$

Solutions to (2.11) afford all possible solutions to (2.1) to (2.3) except...

Equatorial A.1 The Kelvin Wave

For the Kelvin wave $v \equiv 0$, and (2.6) to (2.8) become

$$u_t + \phi_x = 0 \quad (2.12)$$

~~$$\phi_t + u_x = 0$$~~

$$y u + \phi_y = 0 \quad (2.13)$$

$$\phi_t + u_x = 0 \quad (2.14)$$

The general solution to these eqns is

$$u = \phi = F(x \mp t) \exp(\mp y^2/2) \quad (2.15)$$

~~or, dimensionally $u = \phi/c = G(x \mp$~~

The upper solution is required if we are to have finite solutions for large y . Dimensionally (2.15) is

$$u = \phi/c = G(x - ct) \exp\left(-\frac{\beta y^2}{2c}\right) \quad (2.16)$$

The Kelvin wave is semigeostrophic: zonal pressure gradients are balanced by acceleration, while the meridional pressure gradient is balanced by rotation. The Kelvin mode is nondispersive [the group velocity $C_g = C = \text{const} = \sqrt{gH}$].

~~As we will see later,~~

HW: Argue ^{physically} why a westward propagating Kelvin wave is not possible.

Note: the equations (shallow water) (2-6) to (2-8), as we will see, also describe the horizontal structures for ρ in a continuously stratified or layered fluid. In fact, for the ~~o~~ tropical ocean hydrography, these ^{troposphere and atmosphere} ~~stratified~~ eqns ~~describe much of the~~ ^{crudely reproduce} describe the surface circulation after replacing H with the equivalent depth H_e .

~~$5 < c < 2.5 - 2.9$ m/s.~~

	H_e (m)	c (m s ⁻¹)	E folding scale ($\sqrt{\frac{2c}{\rho}}$)
Ocean	0.64	2.5 ~~m/s~~	470 km
Troposphere	3700	60	2200 km

2-A.2 The wave form solution The dispersion relationship

$$T = (c\beta)^{-1/2} \quad L = (c/\rho)^{1/2}$$

1.53 d	332 km
0.31 d	1615 km

Assuming a wave form solution to (2-11)

$$v(y, x, y, t) = \psi(y) \exp(ikx + i\omega t) \quad (2.17)$$

we obtain the well studied ODE

$$\frac{d^2\psi}{dy^2} + \left\{ \omega^2 - k^2 + \frac{k}{\omega} - y^2 \right\} \psi = 0 \quad (2.18)$$

This equation describes a harmonic oscillator with potential $\propto y^2$, is self-adjoint and thus possesses orthogonal eigenfunctions

Solution method: Frobenius + Hermite polynomials.

With $\psi(y)$ of the form

If we write solutions to (2.18) are bounded only
 ψ dispersion relationship

$$y \quad \boxed{\omega^2 - k^2 + \frac{k}{\omega} = 2n+1} \quad n=0, 1, 2, \dots \quad (2.19)$$

and the solution $\psi = \psi_n$ for each n is

$$\psi_n(y) = \frac{\exp(-y^2/2) H_n(y)}{\sqrt{2^n \cdot n!} \sqrt{\pi}} \quad (2.20)$$

where H_n is the n^{th} Hermite polynomial.

The ψ_n are orthonormal:

$$\int_{-\infty}^{\infty} \psi_n \psi_m = \delta_{nm} \quad (2.21)$$

As (2.18) and (2.19) also can be written

$$\frac{d^2}{dy^2} \psi_m + (2n+1 - y^2) \psi_m = 0$$

if $y^2 > 2n+1 \Rightarrow$ exponential decay
 $< 2n+1 \Rightarrow$ oscillatory.

$\therefore y = \sqrt{2n+1}$ is the turning latitude

* Motions are equatorially trapped

2.4.3 Important Relationships (Xerox)

Hermite Polynomial $H_n(y)$

$$H_n(y) = \sum_{s=0}^{n/2} (-1)^s (2y)^{n-2s} \frac{n!}{(n-2s)! s!} \quad (2.22)$$

Rodriguez ~~Form~~ Formula

$$H_n(y) = (-1)^n \exp(y^2) \frac{d^n}{dy^n} (e^{-y^2}) \quad (2.23)$$

Recurrence relations

$$H_{n+1}(y) = 2y H_n(y) - 2n H_{n-1}(y) \quad (2.24)$$

$$\frac{\partial H_n}{\partial y} = 2n H_{n-1}(y)$$

or, in terms of $\psi_n \dots$

$$\frac{d\psi_n}{dy} = -\sqrt{\frac{n+1}{2}} \psi_{n+1} + \sqrt{\frac{n}{2}} \psi_{n-1} \quad (2.25a)$$

$$y \psi_n = \sqrt{\frac{n}{2}} \psi_{n-1} + \sqrt{\frac{n+1}{2}} \psi_{n+1} \quad (2.25b)$$

Hermite Polynomials

$$H_0 = 1$$

$$H_1 = 2y$$

$$H_2 = 4y^2 - 2$$

$$H_3 = 8y^3 - 12y$$

$$H_4 = 16y^4 - 48y^2 + 12$$

$$H_5 = 32y^5 - 160y^3 + 120y$$

2.A.7 The Dispersion Relationship ~~Group Velocity~~

For $k \geq 1$ there are two possible ^{approximate} solutions to (2.19), the dispersion relationship: (third term small) so

$$\Rightarrow \omega^2 = k^2 + 2n + 1$$

which has two solutions

$$\omega = \pm \sqrt{k^2 + 2n + 1} \quad (2.26)$$

corresponding to inertio-gravity waves (- eastw.; + westward)
(2.26) can be written dimensionally as

$$C = \frac{\omega}{k} = \pm \sqrt{gH + \beta \frac{\sqrt{gH}}{k^2} (2n+1)} \quad (2.27)$$

n.b. with $\beta=0$, this ~~is~~ just the shallow water gravity wave
rotation increases C

For small frequency $\omega \sim O(k^{-1})$ (2.19) can be written

$$\omega = k / (k^2 + 2n + 1) \quad (2.28)$$

which, dimensionally, is

$$C = + \beta / (k^2 + \frac{\beta}{\sqrt{gH}} (2n+1)) \quad (2.29)$$

This limit represents the westward propagating Rossby wave. Note the similarity between (2.28)

and the midlatitude equivalent

$$C = \beta / (k^2 + \ell^2 + f^2/gH)$$

$\uparrow$ meridional structure $\uparrow$ divergence term
 $\uparrow$ combined terms in tropics.

2-A.5 The complete dispersion relationship

a) $n=0$ The dispersion relation, with $n=0$, can be factored to yield two solutions

~~$$(k-\omega)(\omega^2 + k\omega - 1)$$~~ (2.30)

The first, $\omega=k$, is unrealistic (why? unbounded solution) and hence discarded. The other roots are

$$\omega = -\frac{k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 + 1} \quad (2.31)$$

correspond to a mixed Rossby-Gravity wave (+)

or a Yanai wave, and an eastward ^{propagating} gravity wave (-)

$$\begin{array}{l} - \\ + \end{array} \left\{ \begin{array}{l} \text{as } k \rightarrow \infty \quad \omega \rightarrow -k, \text{ (kelvin)} \quad \text{negative root.} \\ \text{as } k \rightarrow +\infty \quad \omega \rightarrow 0; \quad k \rightarrow 0 \quad \omega \rightarrow \pm \end{array} \right.$$

Short Rossby Long gravity positive root

b) " $n=-1$ " The kelvin mode

Putting $n=-1$ into (2.19), Matsuno found that the kelvin mode $\omega=k$ is one solution. The kelvin mode meridional structure is $u=\phi=\psi_{n=0}$, however and this nomenclature is just a convenience.



write eqn #s on graph.

2.4.6 Group velocity

Differentiating (2.19) w.r.t. k yields

$$C_g \equiv \frac{\partial \omega}{\partial k} = \frac{2k\omega - 1}{2\omega^2 - k} \quad (2.32)$$

[don't forget in Matsuno's (1968) solution, $C_g < 0$ is eastward group velocity]

So $\omega = (2k)^{-1}$ denotes an extrema (zero) in group velocity. Values for $C_g = 0$ as a function of n are found by plugging $\omega = (2k)^{-1}$ into (2.19)...

$$\omega = \underbrace{\left(\frac{n+1}{2}\right)^{1/2}}_{\uparrow \text{LB}} + \underbrace{\left(\frac{n}{2}\right)^{1/2}}_{\uparrow \text{RW}}, \quad \left(\frac{n+1}{2}\right)^{1/2} - \left(\frac{n}{2}\right)^{1/2}$$

Finally we note for the Rossby waves in the long ~~(2.28)~~ and short wave number limits are group velocity is given by (2.28)

$$\text{long} \quad C_g = \left. \frac{\partial \omega}{\partial k} \right|_{k \rightarrow 0} = \frac{\partial}{\partial k} (\omega = k/(2n+1)) = \frac{1}{2n+1} \quad (\text{westward}) \quad (2.33a)$$

$$\text{short} \quad C_g = \left. \frac{\partial \omega}{\partial k} \right|_{k \rightarrow \infty} = \frac{\partial}{\partial k} (\omega = k^{-1}) = -\frac{1}{k^2} k^{-2} \rightarrow 0 \quad (\text{eastward}) \quad (2.33b)$$

$k \rightarrow \infty$
Large

Finally note that the ~~desper~~ (2.19) can be rewritten

$$k = +\frac{1}{2\omega} \pm \sqrt{\omega^2 + \frac{1}{4\omega^2} - (2n+1)} \quad (2.34)$$

For $\omega^2 + \frac{1}{4}\omega^2 - (2n+1) < 0$, k becomes imaginary
 so there will be westward propagation and ~~decaying~~ ^{evanescent} solutions. (decay or growth). This will become important in a bounded ocean.

2.A.7 The ~~meridional~~ ^(eigenvectors) structure functions for the discrete modes n .

a) $n \geq 1$ meridional mode

~~2.7~~ The meridional ^{velocity} structure for each triad ^{of} frequencies is given by (2.17)

$$v_n(\text{or } x, y, t) = \psi_n(y) \exp(ikx + i\omega t) \quad (2.33)$$

where ψ_n is defined in (2.20). To obtain u , plug (2.17) into (2.10), using recurrence relations (2.23)

$$(k^2 - \omega^2) u_n = + (i\omega y \psi_n + ik \psi_{ny}) \exp(i(kx + \omega t))$$

$$\text{or} \quad = + i \left[\omega (\sqrt{2n} \psi_{n-1} + \sqrt{2(n+1)} \psi_{n+1}) - k (\sqrt{2(n+1)} \psi_{n+1} - \sqrt{2n} \psi_{n-1}) \right] \quad (2.35)$$

$$\Rightarrow u_n = \frac{i}{2} \exp(i(kx + \omega t)) \left\{ \frac{\sqrt{2n} \psi_{n-1}}{\omega - k} + \frac{\sqrt{2(n+1)} \psi_{n+1}}{\omega + k} \right\}$$

To get ϕ , take ∂_x (2.6) and ∂_t (2.8) and subtract

$$\phi_{xx} - \phi_{tt} = y v_x + v_y t \quad (2.37)$$

which yields

$$\phi_n = \frac{i}{2} \exp(i(kx + \omega t)) \left\{ \frac{\sqrt{2n}}{\omega - k} \psi_{n-1} - \frac{\sqrt{2(n+1)}}{\omega + k} \psi_{n+1} \right\} \quad (2.38)$$

For each $n \geq 1$ there are up to three values of ω (see, 2.15; dispersion relationship)
 [limits $\frac{1}{2} \omega$ (large ω); RW small ω]

[Note these ~~become~~ ^{are} Matsunos eqns (17) ~~2.20, 2.21~~.
 To see this multiply (2.35), ~~(2.35)~~ ^(2.37) and (2.37) by $i(\omega^2 - k^2)$
 and note the different normalization in 2.20].

(2.35) - (2.37) will prove easier to work with later on
 when we discuss forced modes.

b) For $n=0$ (Mixed R-G / ~~wave~~ {Yarai Wave})

$$v_0(x, y, t) = \psi_0(y) \exp(i(kx + \omega t)) \quad (2.38)$$

The recurrence relationships, ^(2.20) with $n=0$, yield

$$\frac{d\psi_0}{dy} = \frac{-\psi_1}{\sqrt{2}}; \quad y\psi_0 = \frac{\psi_1}{\sqrt{2}}$$

so the corresponding u, ϕ are

$$u_0 = \phi_0 = \left[\frac{-i\omega}{\sqrt{2}(\omega + k)} \right] \exp(i(kx + \omega t)) \psi_1(y) \quad (2.39)$$

Note for $n=0$ there ~~are~~ ^{are} only two valid solutions to
 the frequency equation (2.31)

$$\omega = -\frac{k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 + 1} \quad (2.31)$$

which is actually 1 continuous mode. To see this

$$\left(\omega + \frac{k}{2}\right)^2 = \left(\sqrt{\left(\frac{k}{2}\right)^2 + 1}\right)^2$$

$$\Rightarrow k = \frac{1}{\omega} - \omega$$

c) $n = -1$ kelvin mode

Using matsuno's normalization ϕ_n , defining $\phi_{n \neq 0} = 0$ allows a convenience. The kelvin mode is, of course,

$$v = 0$$

$$u = \phi = \psi_0 \exp(i(\omega t + kx))$$

where $\omega = -k$ [~~and the constant ψ_0 is arbitrary~~]

[Structure pictures, Matsuno Figs 4-8.]

$EIG \sim \text{like } \kappa \approx 1 \text{ } \underline{\underline{IG}}$

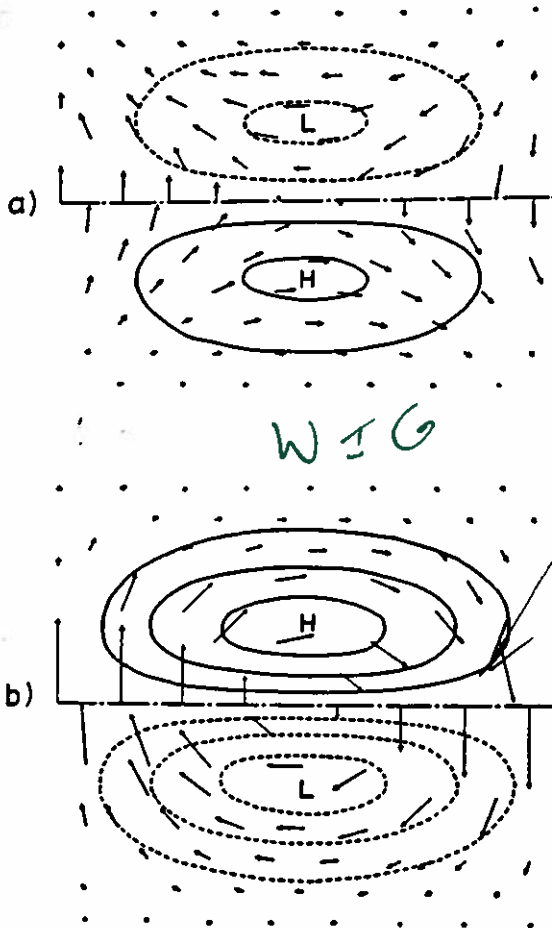


Fig. 6. Pressure and velocity distributions of eigensolutions for $n=0$ and $k=0.5$
 a: Eastward moving inertio-gravity wave
 b: Westward moving inertio-gravity wave.

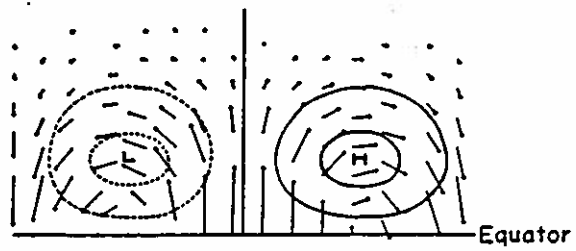


Fig. 7. Rossby type wave ($n=0$ $k=1.0$).

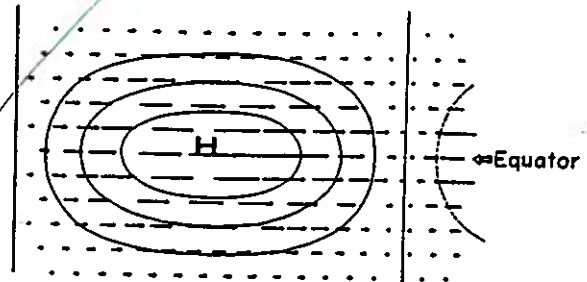


Fig. 8. Pressure and velocity distributions of eigensolution for $n=-1$ and $k=0.5$.
 This wave behaves like as the Kelvin wave.

Agrees, on Eq.

$k \sim 1$ ψ antisym. and u, ϕ sym.

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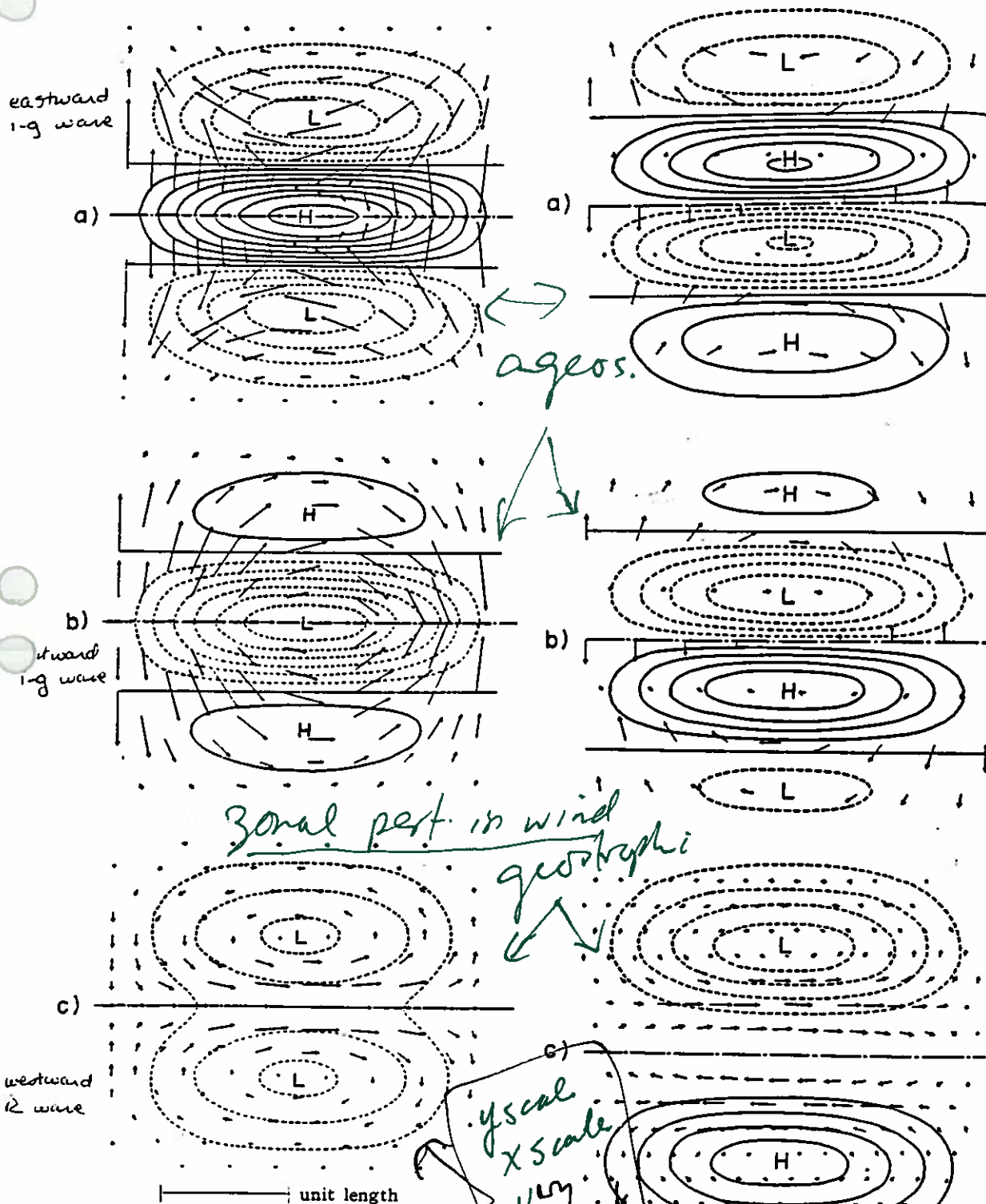


Fig. 4. Pressure and velocity distributions of eigensolutions for $n=1$

- a: Eastward propagating inertio-gravity wave
b: Westward propagating inertio-gravity wave
c: Rossby wave.

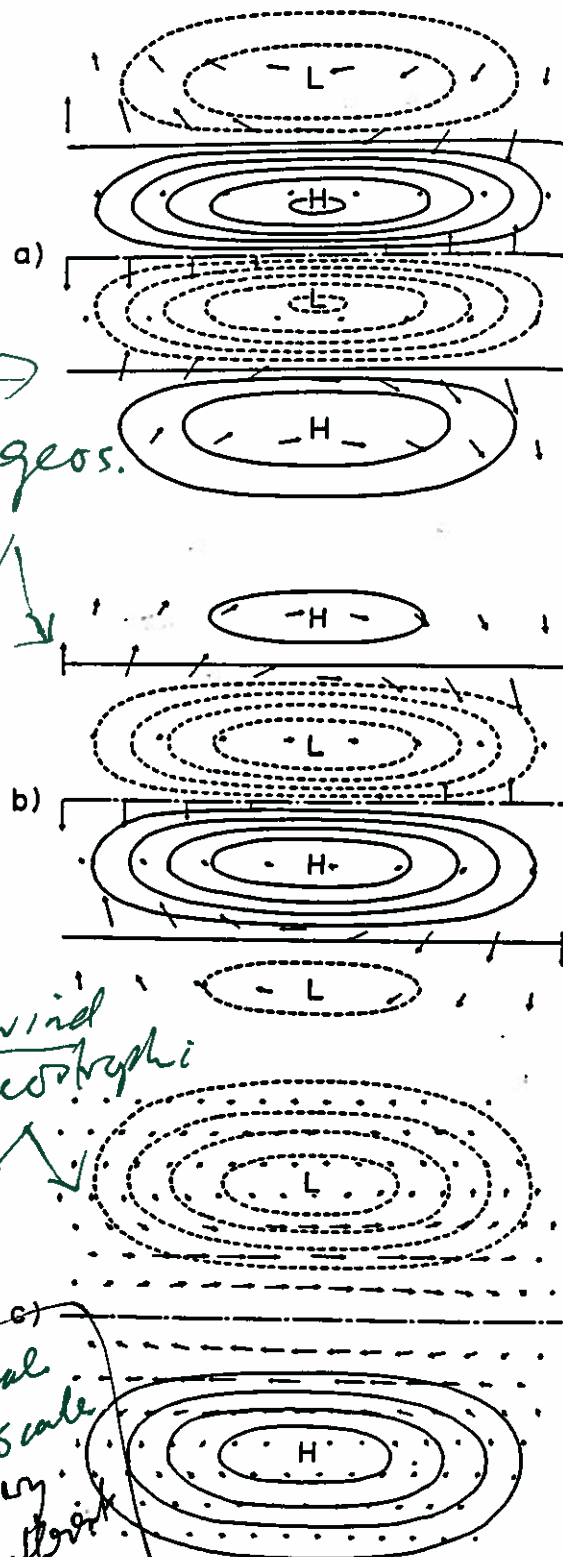


Fig. 5. Same as Fig. 4 but for $n=2$.



Fig. 6. Pressure eigensolutions
a: Eastward
b: Westward

motions in the d

The most inter of Rossby and g modes. Fig. 6a and wind pattern this case, as m solutions, i.e. ea (6b) moving wa moving wave wa in this case fro shown in Fig. 3. moving wave is for $k > 1/\sqrt{2}$ and $1/\sqrt{2}$, from the The pressure and $k=1.0$ are show belongs to the R