

The perturbation pressure gradients are

$$\nabla \tilde{P}^{(1)} = g \nabla \tilde{\eta} \quad (33)$$

and

$$\nabla \tilde{P}^{(2)} = g \left[\nabla \tilde{\eta} + \frac{\Delta \rho}{\rho} \nabla h \right] \quad (34)$$

$$\left[\text{as in eq. } \frac{d\rho}{dz} = -\rho g \right]$$

$$\text{or } \nabla \tilde{P}^{(2)} = g \nabla \tilde{\eta} + g' \nabla h \quad (35)$$

$$\text{where } g' = \frac{(\rho_2 - \rho_1)}{\rho_2} g = \frac{\Delta \rho}{\rho_2} g \approx \frac{1}{200} g \quad (36)$$

We can re-write these equations for the mean depth averaged flow $\tilde{u}_0$ and the sheared flow $\tilde{u}_1$.

$$\mu_1 \times (29) + \mu_2 \times (30) \Rightarrow$$

$$\tilde{u}_{0t} + f \tilde{k} \times \tilde{u}_0 = -\nabla \tilde{P}_0 + \frac{\tilde{\sigma}^s}{\rho_1 H_0} \quad (37)$$

$$\text{Eq (30)} + \text{Eq (32)} \Rightarrow$$

$$\eta_t + H_0 \nabla \cdot \tilde{u}_0 = 0 \quad (38)$$

$$\text{where } P_0 = g\eta + \frac{\mu_2 g' h}{H_0} \approx g\eta \quad (39) \quad \begin{array}{l} \text{since a posteriori} \\ h \approx \eta \end{array}$$

$$\text{and } \tilde{u}_0 = (\mu_1 \tilde{u}^{(1)} + \mu_2 \tilde{u}^{(2)}) / H_0 \quad (40)$$

$$\text{h.b. } C_0 = \sqrt{g H_0} \sim 200 \text{ m/s} \Rightarrow \text{the shallow water barotropic mode}$$

The sheared flow

Eq (29) - Eq (31)

$$\tilde{u}_{1t} + f \underline{k} \times \tilde{u}_1 = -\underline{\nabla} P_1 + \frac{\tilde{\tau}}{\rho_1 H_1} \quad (41)$$

$$P_{1t} + C_1^2 \underline{\nabla} \cdot \tilde{u}_1 = 0 + \text{Order} \left(\frac{H_2 \eta}{H_0 h} \right) \quad (42)$$

$$\text{where } \tilde{u}_1 = \tilde{u}_1^{(1)} - \tilde{u}_1^{(2)} \quad (43)$$

$$\text{and } P_1 = -g' h \quad (44)$$

$$\text{and } C_1^2 = \frac{g' H_1 H_2}{H_0} \ll C_0^2 \quad (45)$$

note: in tropics ~~for 150m~~ $g' = \frac{1}{200} g$; $H_1 = 150\text{m}$ $H_2 \sim 4\text{km}$

$$\Rightarrow C_1 \approx 2.7 \text{ m/s}$$

The 1 1/2 layer model

note for the surface trapped flow $u^{(2)} \rightarrow 0$ & so

$$\nabla P^{(2)} \neq 0, \text{ or using (39)}$$

$$\eta = \frac{-g' H_2 h}{g H_0} = \frac{-\Delta P}{\rho_2} h \ll h \quad (46)$$

So 42 yields

$$P_{1t} + C_1^2 \underline{\nabla} \cdot \tilde{u}_1 = 0 \quad (42')$$

A Eq. 46 says the pycnocline motion is out of phase with the sea level height $h \approx -\rho_2 / \rho_1 h \approx -200 \times \eta$!

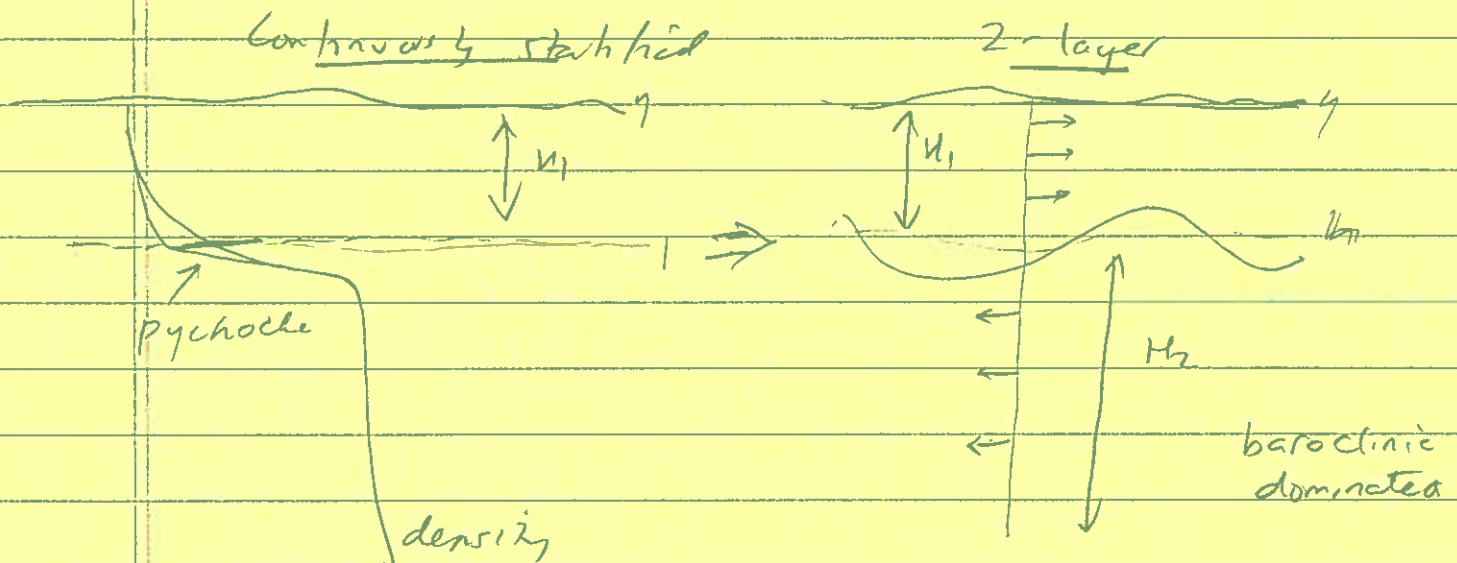
Notes

- The ratio of the forcing of the baroclinic + ~~barotropic~~^{baro} barotropic modes is (cf eqs 41 and 37)

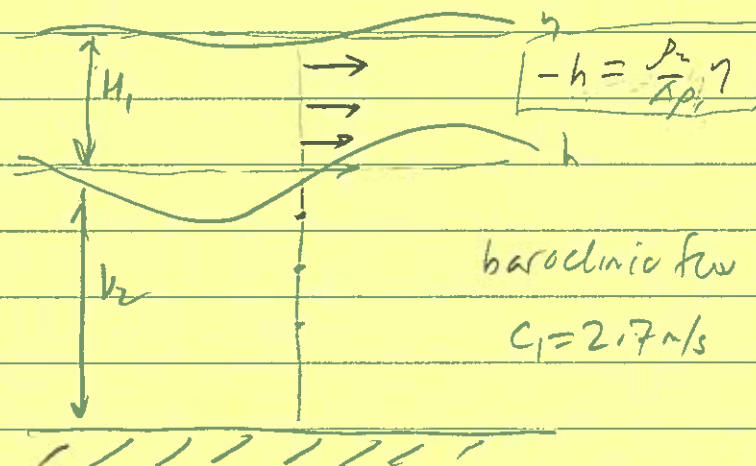
$$\frac{\text{baroclinic}}{\text{barotropic}} \approx \frac{H_0}{H_1} \sim \frac{4 \text{ km}}{200 \text{ m}} \approx 20!$$

- For the case where all the energy,

Hence for interannual motions which have shallow ray paths, we can approximate the continuously stratified ocean by a 2-layer approximation, where the top layer ($\rho_1, H_1 \sim 200m$) is separated from a deep abyssal layer that is motionless ($\rho_2 = \rho_1 + \Delta\rho, H_2 = 4000m$) that is motionless.



1 1/2 layer ($\rho_2 u_2 \equiv 0$)



$$C_1 = 2.7 m/s$$

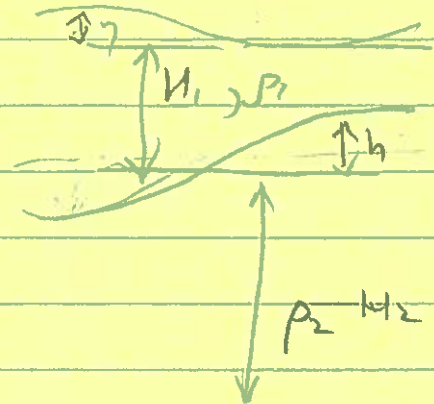
note: 2 1/2 layer approx is better.

not to scale

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Response of the tropical ocean to low frequency
wind forcing

(a) Assume linear, low frequency motions $\Rightarrow$ $1\frac{1}{2}$ layer model



the equations

$$u_t - \beta y v = -P_x + \frac{\tau^x}{\rho_1 H_1} \quad (41a)$$

$$v_t + \beta y u = -P_y + \frac{\tau^y}{\rho_1 H_1} \quad (41b)$$

and

$$P_t + c_1^2 \nabla \cdot \underline{u} = 0 \quad (42')$$

where $c_1^2 = \frac{g' H_1 H_2}{H_1 + H_2} \approx g' H_1$

and $\eta \approx -\frac{\Delta P}{\rho_2} h$

values: $g' = g \frac{\Delta \rho}{\rho_2} = \frac{g}{200}$; $H_1 = 150 \text{ m} \Rightarrow c_1 = 2.7 \text{ m/s}$

(a) The nondimensionalized eqns are (dropping $\sim$):

$$\{ (x, y) = (\tilde{x}, \tilde{y}) (\frac{c}{\beta})^{1/2}; \quad t = \tilde{t} (\frac{c}{\beta})^{1/2}; \quad u = \tilde{u} c; \quad P = \tilde{P} c; \}$$

$$u_t - yv = -P_x + F^x \quad (47)$$

$$v_t + yu = -P_y + F^y \quad (48)$$

$$\text{or } P_t + u_x + v_y = 0, \quad (49)$$

$$\text{where } \underline{F} = (F^x, F^y) = \frac{\tau}{\rho_1 u_1} \cdot \left(\frac{c}{\beta} \right)^{-3/2} \quad (50)$$

(b) The longwave approximation.

Take $\left(\frac{\partial}{\partial t} \text{ of } 47 \right) + (y \times 48)$ to obtain

$$\left(\frac{\partial^2}{\partial t^2} + y^2 \right) u = -y P_y - P_{xt} + y F^y + F_t^x \quad (51)$$

We are interested in low frequency solutions

where

$$\left| \frac{\partial}{\partial t} \right| = |\omega| \ll |y| \quad \text{and} \quad (52)$$

and since the ~~zonal~~ wind's zonal scale is much larger than the oceanic meridional

scale,

$$\left| \frac{\partial^2}{\partial x \partial t} \right| \ll \left| y \frac{\partial}{\partial y} \right| \sim O(1). \quad (53)$$

Under approximations 52 and 53, eqn 51 reduces to the "forced geostrophic" balance

$$\boxed{y u = -p_y + F^y} \quad (54)$$

(cf eqn. 48). Effectively we have eliminated the IG waves, the MRG (Yanai) wave and all the short Rossby waves (show dispersion again)

c) Solutions

We could solve 47, 54 and 49 using Fourier expansion (e.g. Matsuno) but ocean boundaries are a problem. Instead we follow Gill (1980) by writing

$$q = p + u, \quad (55)$$

$$r = p - u \quad (56)$$

Now take (47 - 49) and use (56) to get

$$r_t - r_x + v_y - yv = -F^x \quad (57)$$

Similarly take (47 + 49) and use (55) to get

(22)

$$q_t + q_x + v_y - yv = F^x \quad (58)$$

Now since

$$p = \frac{1}{2}(q+r) ; \quad u = \frac{1}{2}(q-r), \quad (59)$$

eqn (54) can be written

$$yq - yr + q_y + r_y = 2F^y \quad (60)$$

Recall our mentioned functions $\psi_m = \psi_m(x, y)$

$$\psi_m = \frac{\exp(-y^2/2) H_m(y)}{\sqrt{2^m m! \sqrt{\pi}}} ; \quad \int_{-\infty}^{\infty} \psi_m \psi_n dy = \delta_{nm} \quad (61)$$

where $H_m(y)$ is the m^{th} Hermite polynomial.

ψ_m are a complete orthonormal set, Expand

our variables q, r , and v in terms of ψ_m

$$\begin{Bmatrix} q(x, y, t) \\ r(x, y, t) \\ v(x, y, t) \end{Bmatrix} = \sum_{m=0}^{\infty} \begin{Bmatrix} q_m(x, t) \psi_m(y) \\ r_m(x, t) \psi_m(y) \\ v_m(x, t) \psi_m(y) \end{Bmatrix} \quad (62)$$

Plugging into (57) we obtain

$$\sum_{m=0}^{\infty} (r_{mt} - r_{mx}) \psi_m + v_n \left(\frac{d\psi_m}{dy} + y \psi_m \right) = -F^x \quad (63)$$

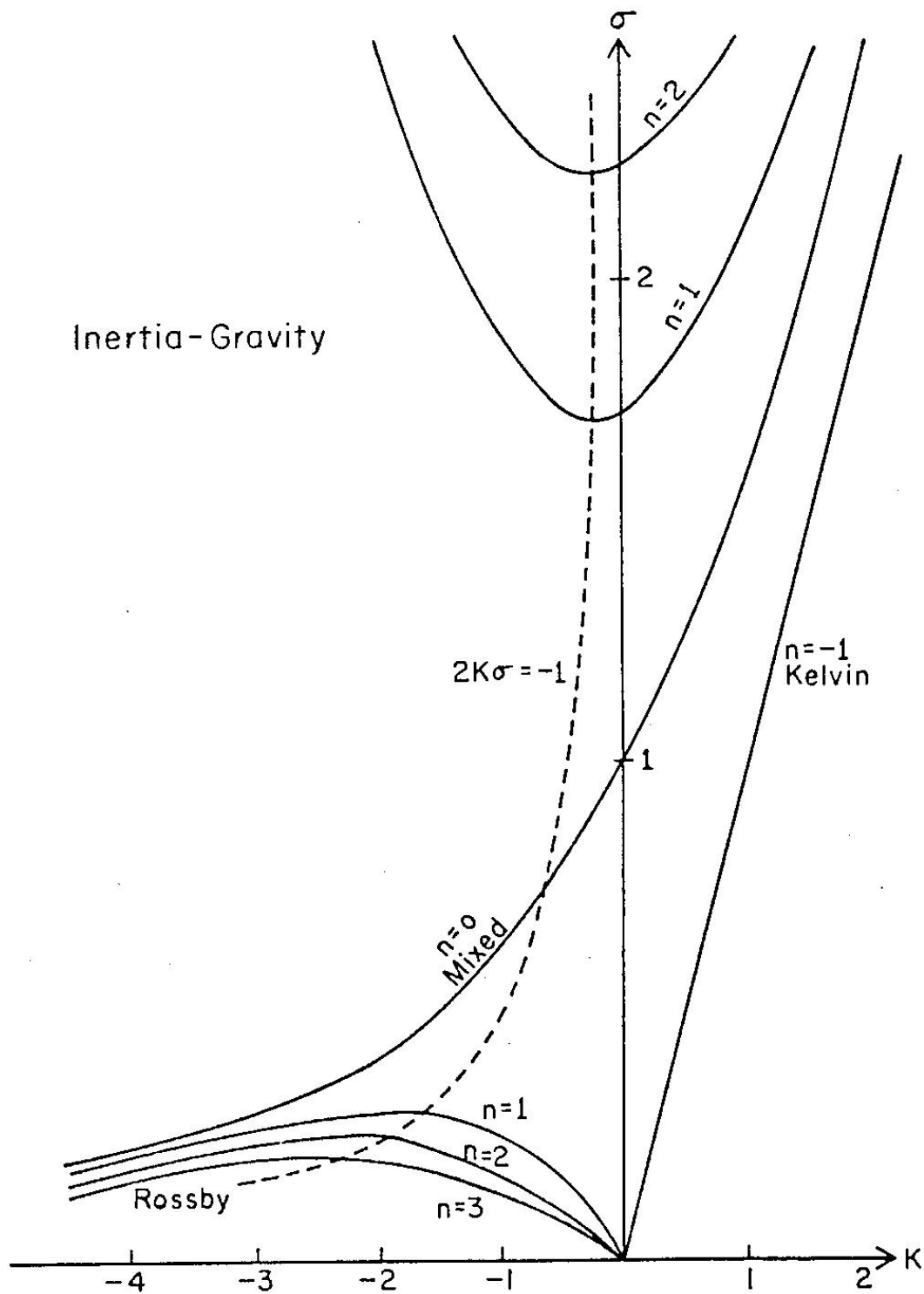


Figure 3.8. Dispersion diagram for equatorially trapped modes. The unit of frequency is $(\beta c)^{1/2}$ and the unit of zonal wave number k is the inverse of the radius of deformation $(c/\beta)^{1/2}$. [From Cane and Sarachik (1976).]

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Multiplying (63) by ψ_0 , and recalling the recursion relationship

$$\frac{d\psi_m}{dy} + y\psi_m = \begin{cases} \sqrt{2m}\psi_{m-1} \\ 0 \end{cases} \quad m=0 \quad (64)$$

We can integrate $\int_{-\infty}^{\infty} (63) \psi_0 dy$ and obtain

$$r_{m+1} - r_{mx} + \sqrt{2(m+1)} \psi_{m+1} = - \int_{-\infty}^{\infty} \psi_m F^x dy \quad (65)$$

$m=0, 1, 2, \dots$

The same manipulation on (58) yields

$$q_{m+1} + q_{mx} - \sqrt{2m} \psi_{m-1} = \int_{-\infty}^{\infty} \psi_m F^x dy \quad (66a)$$

$$q_{0+1} + q_{0x} = \int_{-\infty}^{\infty} \psi_0 F^x dy \quad (66b)$$

$m=0$

Expansion of (60) yields

$$q_{m+1} \sqrt{2(m+1)} - \sqrt{2m} r_{m-1} = 2 \int_{-\infty}^{\infty} \psi_m F^y dy \quad (67a)$$

$m \neq 0$

$$q_1 = \sqrt{2} \int_{-\infty}^{\infty} \psi_0 F^y dy \quad m=0 \quad (67b)$$

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We can further simplify by elimination v_m from (67a) and (65) to obtain

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) q_{m+1} + \frac{\sqrt{2} v_m' m}{\sqrt{m+1}} &= \sqrt{\frac{2}{m+1}} \int_{-\infty}^{\infty} (F_t^y - F_x^y) \psi_m dy \\ &- \sqrt{\frac{m}{m+1}} \int_{-\infty}^{\infty} F^x \psi_{m-1} dy \end{aligned} \quad (68)$$

$m=1, 2, \dots$

Replacing m in (66a) by $m+1$, we can eliminate v_m in (68):

$$\begin{aligned} (2m+1) \frac{\partial}{\partial t} q_{m+1} - \frac{\partial}{\partial x} q_{m+1} &= \int_{-\infty}^{\infty} F^x \left[m \psi_{m+1} - \sqrt{m(m+1)} \psi_m \right] dy \\ &+ \sqrt{2(m+1)} \int_{-\infty}^{\infty} (F_t^y - F_x^y) \psi_m dy \end{aligned} \quad (69)$$

$m=1, 2, \dots$

So all the q_m are known in terms of the forces

(q_0 and q_1 are given by 66b and 67b) - The v_m are then also known, by 67a. Hence all the p_m and u_m are known. To get v_m , use (66a, b) and (69):

$$\begin{aligned} v_m &= \sqrt{2(m+1)} \frac{\partial}{\partial t} q_{m+1} + \int_{-\infty}^{\infty} F^x \left[\frac{\sqrt{m}}{2} \psi_{m-1} - \sqrt{\frac{m+1}{2}} \psi_{m+1} \right] dy \\ &+ \int_{-\infty}^{\infty} (F_x^y - F_t^y) \psi_m dy \end{aligned} \quad (70a)$$

ad

$$v_0 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) q_1 - \int_{-\infty}^{\infty} F^x \psi_1 dy \quad (70b)$$

(d) Lowest order solutions

Since for low frequency motions the forcing scales give $|\frac{\partial}{\partial t}, \frac{\partial}{\partial x}| \ll 1$, the eqns (66b),

(66b, 67b, 69), 70a,b
 q v in q, q

$$(66b) \Rightarrow q_{0t} + q_{0x} = - \int_{-\infty}^{\infty} F^x \psi_0 dy \quad (71a)$$

$$(67b) + (66b) \quad |q_1| \sim \left| \int_{-\infty}^{\infty} \psi_0 F^y dy \right| \sim |q_{0t} + q_{0x}| \sim O\left(\frac{q_0}{x_{\text{scale}}}\right) \ll 1 \quad (71b)$$

$$(69) \Rightarrow (2m+1) \frac{\partial}{\partial t} q_{m+1} - \frac{\partial}{\partial x} q_{m+1} = \int_{-\infty}^{\infty} F^x \left[m \psi_{n+1} - \sqrt{m(m+1)} \psi_{n-1} \right] dy \quad m=1, 2, \dots \quad (71c)$$

$$67a \quad r_{m-1} = \sqrt{\frac{m+1}{m}} q_{m+1} \quad m=1, 2, 3, \dots \quad (71d)$$

$$v_m = \sqrt{2(m+1)} \frac{\partial}{\partial t} q_{m+1} + \int_{-\infty}^{\infty} F^x \left[\sqrt{\frac{m}{2}} \psi_{m-1} - \sqrt{\frac{m+1}{2}} \psi_{m+1} \right] dy \quad m=1, 2, \dots \quad (71e)$$

$$(70b) \quad v_0 = -\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} F^x \psi_1 dy \quad (71d)$$