

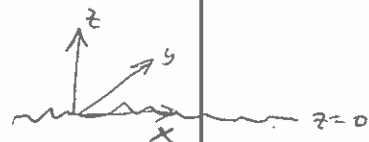
Normal Modes in a continuously stratified ocean.

①

Refs: See attached sheet; also Cane (1984, JPO);
McCreary (1981, Phil. Trans.); Gill (1982, Academic);
Eriksen (1982, JPO); Bigg and Blundell (1989, QJRM);
Gill and Clarke (1977, DSR);

Assume Rossby No. $Ro \ll 1$. Boussinesq approximation
Ignore dissipation for now. Assume driven by wind
stresses, τ_x , only.

Momentum eqns.



$$\underline{u}_t + f \underline{k} \times \underline{u} = -\frac{\nabla P}{\rho_0} + \frac{\underline{\tau}}{\rho_0} \quad (1)$$

where $\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j}$ on the equatorial β -plane,
 $\underline{u} = u \hat{i} + v \hat{j}$, and ρ_0 is a constant reference
density, related to the total density ρ by...

$$\rho = \rho_0 + \bar{\rho}(z) + \rho_0 \sigma(x, y, z, t) \equiv \rho_B(z) + \sigma \quad (2)$$

where $|\sigma|, |\bar{\rho}/\rho_0| \ll 1$.
Write the total pressure P such that

$$P = P_B(z) + \rho_0 \tilde{P}(x, y, z, t) \quad (3)$$

The hydrostatic balance then requires that

$$\frac{dP_B}{dz} = -\rho_B g \quad ; \quad \text{and} \quad \boxed{\frac{\partial \tilde{P}}{\partial z} = -\frac{\sigma g}{\rho_0}} \quad (4)$$

and, in eqn ① $\frac{\nabla p}{\rho_0} = \nabla \tilde{p}$.

Continuity (incompressible)

$$\boxed{\nabla \cdot \tilde{u} + w_z = 0} \quad (5)$$

and

$$\frac{Dp}{Dt} = 0 \text{ yields}$$

$$\tau_t = -\tilde{u} \cdot \nabla p - w \frac{\partial p}{\partial z}$$

or, linearizing

$$\tau_t = -w \frac{dp_B}{dz} \quad (6)$$

$$\text{or } \boxed{\frac{\tau_t}{\rho_0} = \frac{N^2 w}{g}} \quad (7) \text{ where}$$

$$\boxed{N^2 \equiv -\frac{g}{\rho_0} \frac{dp_B}{dz}} \quad (8)$$

Eqs ①, ④, ⑤ and ⑦ are 5 eqs w/ 5 unknowns:

Assume the unknowns of the form:

$$\left\{ \begin{array}{c} \tilde{u} \\ \tilde{p} \\ w \\ \tau \\ \left(\frac{\tilde{z}}{\rho_0} \right) \end{array} \right\} = \sum_{m=0}^{\infty} \left\{ \begin{array}{c} \tilde{u}_m(x, y, t) \quad F_m(z) \\ \tilde{p}_m(x, y, t) \quad F_m(z) \\ w_m(x, y, t) \quad \int_{-H}^0 F_m(z) \\ \tau_m(x, y, t) \quad F_{mz}(z) \\ \tilde{\tau}_s(x, y, t) \quad X(z) \end{array} \right\} \quad (9)$$

where $F_m(z)$ will be the vertical normal modes that satisfy the appropriate Sturm-Liouville problem.

Substituting (9) into (1) and (5) yields

$$\underline{u}_m + f \underline{k} \times \underline{u}_m = -\nabla P_m + \frac{\tilde{\tau}_s \int_{-H}^0 \underline{x} F_m dz}{\int_{-H}^0 F_m^2 dz} \quad (10)$$

$$\nabla \cdot \underline{u}_m + W_m = 0 \quad (11)$$

Combining (4) and (7) we obtain

$$\tilde{P}_{zt} = -\frac{g}{\rho_0} \tau_t = -N^2 W \quad (12)$$

Or, substituting from (1) --

$$\frac{-F_m z}{N^2 \int_{-H}^z F_m dz} = \frac{W_m(x, y, t)}{P_{mt}(x, y, t)} = \text{const} \equiv \frac{1}{C_m^2} \quad (13)$$

So (13) and (11) yield

$$\frac{P_{mt}}{C_m^2} + \nabla \cdot \underline{u}_m = 0 \quad (14)$$

[does this look familiar?]

and the F_m must satisfy

$$\left(\frac{F_m z}{N^2} \right)_z + \frac{F_m}{C_m^2} = 0 \quad (15)$$

under the appropriate boundary conditions (B.C.)

Notes:

- ① Eqs (10) and (14) are just the shallow water eqns. with equivalent depth given by

$$H_e = \frac{C_m^2}{g}$$

where the C_m are determined through (15) w/ B.C.

The vertical structure is F_n .

The boundary conditions

- (a) Flat bottom at $z = -H$, so $W(-H) = 0$

$$\text{By (7)} \quad W \propto \sigma_t = \sum_{n=0}^{\infty} \sigma_{tm} \cdot F_m z$$

$$\Rightarrow \boxed{F_{mz} = 0 \text{ at } z = -H} \quad (16)$$

- (b) At the surface $z = \eta$, where η is assumed to be a "small" perturbation about rest

$$\frac{D}{Dt}(z - \eta) = 0 \quad \text{on } z = \eta$$

(definition of surface). Linearized, this becomes

$$W = \eta_t \quad \text{on } z = \eta \quad (17)$$

But, using the hydrostatic balance

$$\int_{z=0}^{\eta} \rho dz = p(\eta) - p(0) = -g \int_0^{\eta} \rho dz \approx -g \rho_0 \eta$$

Now $P(\eta)$ is the sea level pressure, assumed constant
so we find

$$-P_t(0) = -g\rho_0\eta_t \quad \text{or} \quad \eta_t = \frac{P_t(0)}{g\rho_0} = \frac{\tilde{P}_t(0)}{g} \quad \text{on } z=0$$

So the free surface B.C. ⁽¹⁷⁾ can be written

$$W(\eta) \begin{cases} \tilde{w}(0) = \frac{-\tilde{P}_t}{N^2} \\ \tilde{w}(0) = \frac{-\tilde{P}_t}{N^2} = \eta_t = \frac{\tilde{P}_t}{g} \quad \text{on } z=0 \\ \text{using (12)} \\ = \eta_t = \frac{\tilde{P}_t}{g} \end{cases}$$

In terms of the F_m , this is expressed (see 9):

$$\boxed{\frac{F_m z}{N^2} + \frac{F_m}{g} = 0} \quad \text{on } z=0 \quad (18)$$

$$\boxed{F_m z = 0} \quad \text{on } z=-H \quad (19)$$

Eqn (15) w/ B.C. (18) and (19) define the orthogonal F_m .

One customary approach is to normalize the eigenvectors F_m

$$\text{by} \quad \int_{-H}^0 F_n F_m dz = \begin{cases} H & n=m \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

where H is the ocean depth.

Notes:

① With the free surface B.C., (15) can be integrated
and, using (18) we find

$$\frac{F_m(0)}{g} = \frac{1}{C_m^2} \int_{-H}^0 F_m dz \quad (21)$$

(2) i.e. The rigid lid B.C. at $z=0$ would be $F_{mz}=0$ on $z=0$.

(3) Special Case: $m=0$

Since $N(z)$ is small everywhere compared to at $z=\eta$, the A/O interface, one mode exists such that (see (15)) $F(z) = \text{const.}$

By (20) we find $\boxed{F_{m=0}(z) = 1}$ and, using

$$(21) \quad \frac{1}{g} = \frac{1}{C_0^2} H \quad \text{or} \quad C_0 = \sqrt{gH}.$$

The barotropic mode.

(4) For a rigid lid and constant stratification N^2 , simple eigenmodes exist: $1, 2, 3, \dots$

$$F_m = A \cos\left(\frac{N}{C_m} z\right)$$

Using the B.C. $F_{mz}=0$ on $z=0, -H$ and (20) we find

$$F_m = \sqrt{2} \cos\left(\frac{m\pi z}{H}\right)$$

$$\text{and} \quad C_m = \frac{NH}{m\pi} \quad m=1, 2, 3, \dots$$

If we take a typical depth averaged value for N
 $g \quad N = 2 \times 10^{-3} \text{ s}^{-1} \quad \text{and} \quad H = 5000 \text{ m}$

$$\Rightarrow C_m \approx \frac{3}{m} \text{ m/s.}$$

- ⑤ Actual Structure using realistic stratification $N^2(z)$
 From Eriksen (1992)

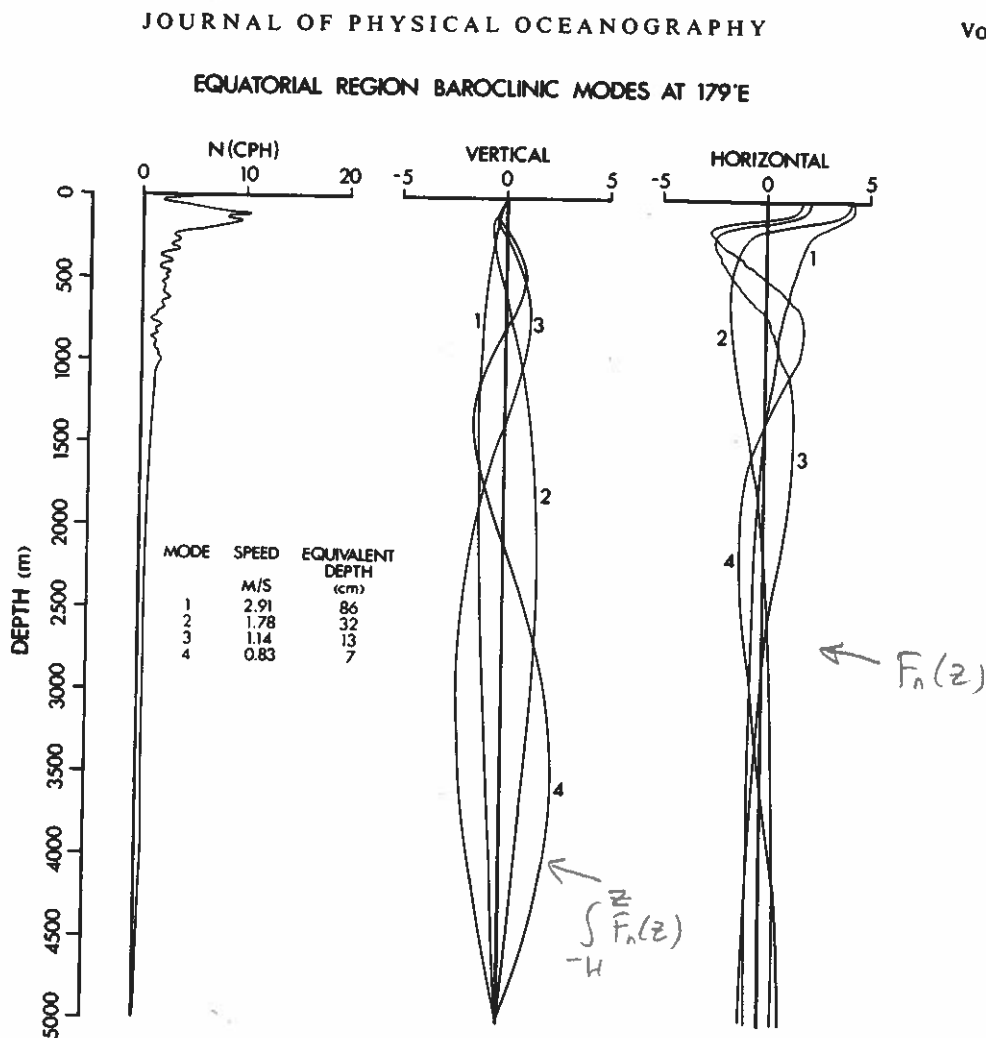


FIG. 4. Profiles of buoyancy frequency N and the first four baroclinic-mode shapes of vertical displacement and horizontal current computed from average $N^2(z)$ profiles between 3°N and 30°S along 179°E.

These are not sinusoidal (N^2 varies strongly).

The fact that $F_m = \text{const}$ in the near surface reflects that this is a well mixed region.

The wind forcing

Using ⑨, the momentum forcing $\frac{\hat{\tau}_z}{\rho_0}$ is distributed into each mode m . The degree of forcing of each