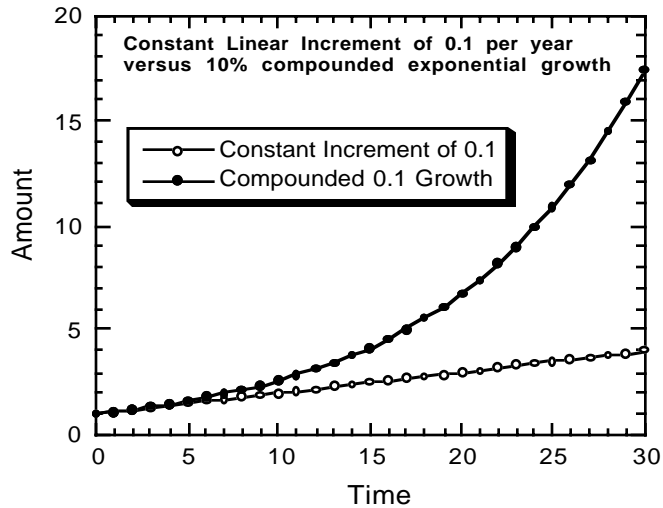


Exponential Growth:

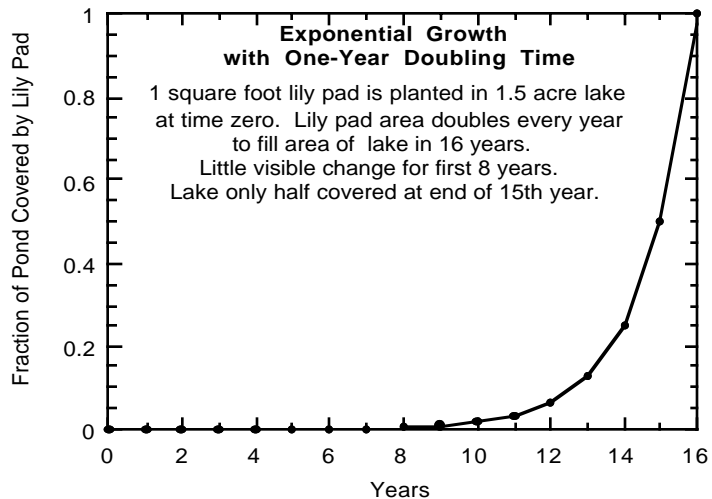
1) Compound Versus Simple Interest

Suppose you put a dollar in an investment that pays you 10% every year and then you put the ten cents in your mattress. How much money do you have in 30 years? About \$4. What if, instead, you put the 10% back into the investment every year so that your gains compound? At the end of 30 years you have about \$17.45, a difference of about 548% in your gain.



2) The Pond and the Lily Pad Problem

Suppose you have a 1.5 acre (65,536 sq.ft.) fish pond on your farm. You want to improve the looks of your pond and the habitat for your trout, so you plant a 1 square foot patch of pond lily. A year later you notice that the area of the pond lily has doubled. At this rate, how long before the pond is covered with lily pads, and how much of the pond is covered the year before total coverage is achieved? It takes 15 years to cover half the pond with lily pads, but one year later at the end of the 16th year all the pond is covered. For the first 8 years, a negligibly small fraction of the pond is covered. It takes over 12 years to cover 10% of the pond.



Lessons:

- Compound interest pays big dividends. The first dollar you save is the biggest.
- Even though the growth is a steady rate, like 10% or 100% per year as in these examples, the magnitude of the year-to-year change increases with time.
- If you are approaching an upper limit exponentially, that limit is approached very quickly after a relatively long wait. For a long time it seems like nothing is happening, then suddenly change is rapid and it's over.

Mathematics Side Bar:

When you calculate the future value of an investment or something else that is growing on a percentage basis, you end up with formulas with powers in them. These can all be converted to

exponential formulas using logarithms and exponents. This section is for those of you who learned exponential functions, but never though they were useful.

As shown in class, if you invest a dollar at 10 percent interest, compounded yearly, then at the end of n years, where n is an integer, your dollar will grow to be, $(1.1)^n$. That is, for every year that passes you multiply what you had last year by the factor 1.1, that is the 1.0 you had at the beginning of the year, plus the 0.1 you get in interest. They multiply, rather than add, because you get to collect interest on last year's interest.

If you started with some arbitrary initial amount of money x_0 , then the money you would have after n compounding periods at the interest rate of ten percent, would be x_n , where,

$x_n = x_0 \cdot (1.1)^n$. Lets generalize this for an arbitrary interest rate and define

$A = 1.0 + \text{fractional gain in one compounding period}$. Then our formula for the amount after n

compounding periods is $x_n = x_0 \cdot A^n$. It turns out that you can write relationships of this form as exponential functions, if you know how to use logarithms and exponentials. Start with the recursive relationship derived above,

$$x_n = x_0 \cdot A^n \quad (1)$$

and rearrange it a little,

$$\frac{x_n}{x_0} = A^n \quad (2)$$

Now you can see from (2) that A^n is the amplification factor, the ratio of the final to the original amount. Next take the natural logarithm of both sides of (2).

$$\ln\left(\frac{x_n}{x_0}\right) = \ln(A^n) = n \ln(A) \quad (3)$$

If we now exponentiate both sides of (3), we get,

$$\left(\frac{x_n}{x_0}\right) = e^{n \ln(A)}$$

or,

$$x_n = x_0 e^{n \ln(A)} \quad (4)$$

Now, since we know what result we want to get to, we can define,

$$\tau = \frac{\Delta t}{\ln(A)}, \quad (5)$$

where Δt is the time interval for compounding and over which our amount grows by the factor A . τ has units of time, and so is a time scale. Putting (5) into (4) we get,

$$x_n = x_0 e^{(n \Delta t / \tau)} \quad (6)$$

So for every interval of time, τ , our amount increases by a factor of $e = 2.71828$. We can call this the e-folding time, the time it takes our amount to increase by the factor $e = 2.71828$. It is obviously a little longer than the doubling time. For example, if the doubling time is one compounding interval, then the e-folding time is,

$$\tau = \frac{\Delta t}{\ln(A)} = \frac{1}{\ln 2} = 1.44 \text{ compounding intervals} \quad (7)$$

Or, if the interest rate is 10% per compounding interval, then the e-folding time is,

$$\tau = \frac{\Delta t}{\ln(A)} = \frac{1}{\ln(1.1)} = 10.49 \text{ compounding intervals. The doubling time } \tau_2 \text{ is related to the e-}$$

folding time τ by the relationship $\tau_2 = \tau \cdot \frac{\ln(A)}{\log_2(A)}$, where \log_2 is a base-two logarithm.