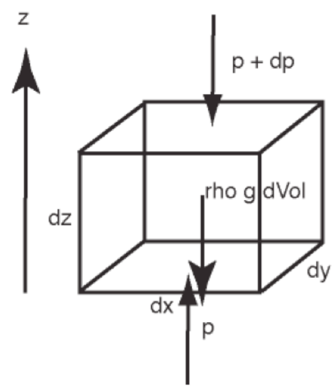
The Hydrostatic equation.

Consider a rectangle of air with dimensions dx, dy and dz, where these are very small increments in the eastward, northward and upward directions, as pictured below. The area of the horizontal surfaces is dA = dx dy.



We consider the force balance in the vertical, positive z, direction.

On the bottom we push up with the force p dA, where p is the pressure in Newtons per square meter, on the top we push downward with force (p + dp) dA. The force of gravity pulls downward with force ρ g dz dA, where dz dA = dVolume, so that ρ dz dA equals the mass of air or water in the rectangle (control volume).

The force balance in the z direction is then,

$$p\, dA - (p + dp)\, dA - \rho\, g\, dz\, dA = 0$$

divide through by dA and use p – p = 0, to get,

$$- dp - \rho\, g\, dz = 0$$

or $dp = - \rho\, g\, dz$

This means that the pressure change must be negative, or the pressure must decrease with height.

We can use the ideal gas law, $p = \rho R T$, where T is the temperature in Kelvins and R is the gas constant, to write,

$$dp = - (p/(RT))\, g\, dz$$

Assuming that T is constant, isothermal, and defining H = RT/g,

$$dp/p = - dz/H \quad \text{or} \quad d\ln p = -dz/H$$

We can then integrate from the surface where $p=p_o$ and z=0, in this way,

$$\int_{p_o}^{p} d\ln p = -\frac{1}{H}\int_{0}^{z} dz$$

Integrating, we get,

$$\ln p - \ln p_o = -\frac{z}{H} \quad \text{or} \quad \ln\left(\frac{p}{p_o}\right) = -\frac{z}{H}$$

Then exponentiating each side of the equation and using $e^{\ln x} = x$, we get

$$p = p_o\, e^{-z/H}$$

a simple exponential dependence of pressure on height, which means that pressure decreases by a factor of $e^{-1} = 0.3679$ every scale height we go upward. A scale height for a temperature of 260K is about 7.6km, if R = 287 J kg^{-1} K^{-1}.

Example: How does the pressure at a fixed elevation change, if the atmosphere warms up?

If the atmosphere warms up, the scale height increases, $H = \frac{RT}{g}$, so that one must go to a greater altitude to reach the same pressure drop from the surface. The atmosphere expands as it warms up, so some mass moves above a fixed altitude as the atmosphere warms.

Let's calculate the pressure increase at 5km associated with a 5C increase in atmospheric temperature.

$$p_2 - p_1 = p_o\left(e^{-z/H_2} - e^{-z/H_1}\right)$$

T_1 = 260K, T_2 = 265K, or H_1 =7,600m, H_2 = 7753m.

$$p_2 - p_1 = 101325Pa\left(e^{-5000/7753} - e^{-5000/7600}\right) = 685Pa = 6.85mb$$