Generalizing the Boussinesq Approximation to Stratified Compressible Flow

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Abstract

The simplifications required to apply the Boussinesq approximation to compressible flow are compared with those in an incompressible fluid. The larger degree of approximation required to describe mass conservation in a stratified compressible fluid with the Boussinesq continuity equation has led to the development of several different sets of "anelastic" equations that may be regarded as generalizations of the original Boussinesq approximation. These anelastic systems filter sound waves while allowing a more accurate representation of non-acoustic perturbations in compressible flows than can be obtained using the Boussinesq system. The energy conservation properties of several anelastic systems are compared under the assumption that the perturbations of the thermodynamic variables about a hydrostatically balanced reference state are small. The "pseudo-incompressible" system is shown to conserve total kinetic and anelastic dry static energy without requiring modification to any governing equation except the mass continuity equation. In contrast, other energy conservative anelastic systems also require additional approximations in other governing equations. The pseudo-incompressible system includes the effects of temperature changes on the density in the mass conservation equation, whereas this effect is neglected in other anelastic systems. A generalization of the pseudo-incompressible equation is presented and compared with the diagnostic continuity equation for quasi-hydrostatic flow in a transformed coordinate system in which the vertical coordinate is solely a function of pressure.

1. Introduction

The Boussinesq approximation is often used to simplify the equations governing fluid motion in order to facilitate both theoretical analysis and numerical computation. Different challenges are encountered when justifying the Boussinesq approximation for liquids and for gases because liquids are nearly incompressible, whereas gases are not. If the fluid is incompressible (i.e., if density is independent of pressure) and if the flow is adiabatic, the only way to change the density of a fluid parcel is through the diffusion of heat and

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trace constituents through the sides of the parcel, and in most applications the influence of such diffusion on density can be neglected [2, p. 75]. Thus, in adiabatic incompressible flow,

$$\frac{D\rho}{Dt} = 0. \tag{1}$$

Here D/Dt denotes the convective derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \tag{2}$$

and $\mathbf{u}$ is the velocity vector.

Using (1), the mass continuity equation

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0. \tag{3}$$

reduces to the simple requirement of nondivergent flow,

$$\nabla \cdot \mathbf{u} = 0. \tag{4}$$

Replacing the prognostic continuity equation (3) with the diagnostic condition (4) eliminates sound-wave (or elastic-wave) solutions from the governing equations and thereby allows the approximate governing equations to be integrated numerically using explicit time-differencing with a much larger time step than that required to preserve stability during integrations of the unapproximated system.

It can also be advantageous to eliminate sound waves from the approximate equations governing motions in gases in cases where such waves are physically insignificant, but still impose a highly restrictive limitation on the maximum time step that may be used to integrate the full compressible equations. Simply requiring the flow to be nondivergent is, however, a relatively crude way to eliminate sound waves from stratified compressible fluids. In particular, (4) will not provide a good approximation to (3) unless the depth of the flow is much shallower than the scale height of the undisturbed density field [15], but this is not the case for many types of atmospheric motions.

Considerable effort has therefore been devoted to deriving generalized Boussinesq equations that eliminate sound waves while more accurately representing the behavior of the other low-Mach-number [1] dynamical modes supported by the exact compressible equations. A major focus of such *anelastic* approximations involves relaxing the assumption of nondivergent flow to obtain a better approximation to true mass conservation. Since the vertical gradient of the reference-state density can be large, many anelastic systems approximate the mass continuity equation as

$$\frac{w}{\overline{\rho}}\frac{d\overline{\rho}}{dz} + \nabla \cdot \mathbf{u} = 0, \tag{5}$$

where $\overline{\rho}(z)$ is the vertically varying density in the reference state and w is the vertical component of the velocity [14,11]. When using this approach, some care is required in the approximation of the pressure gradient terms in the momentum equations to ensure that the resulting system is energy conservative [10].

An alternative approach suggested by Durran [5] is to define a pseudo-density ρ^* and to enforce mass conservation with respect to this pseudo-density such that

$$\frac{1}{\rho^*}\frac{D\rho^*}{Dt} + \nabla \cdot \mathbf{u} = 0. \tag{6}$$

[1] The Mach number is the characteristic velocity scale divided by the speed of sound.

In the case of perfect gases, the pseudo-density may be defined as $\rho^* = \overline{\rho}\overline{\theta}/\theta$, where $\theta = T\pi^{-1}$ is the potential temperature, T is the temperature, $\pi = (p/p_s)^{R/c_p}$, c_p is the specific heat at constant pressure, R is the gas constant, p_s is a constant reference pressure, and $\overline{\theta}(z)$ is the potential temperature of the reference state. Using this definition for ρ^*, (6) may be combined with the *unapproximated* momentum and thermodynamic equations to yield an energy-conservative "pseudo-incompressible system" that does not support sound waves. Note that using the thermodynamic equation

$$\frac{DS}{Dt} = \frac{D}{Dt}\left(c_p \ln\theta\right) = \frac{Q}{T}, \tag{7}$$

where S is entropy and Q is the heating rate per unit mass, (6) may be alternatively expressed as the diagnostic "pseudo-incompressible" equation

$$\nabla \cdot \left(\overline{\rho}\overline{\theta}\mathbf{u}\right) = \frac{\overline{\rho}Q}{c_p\overline{\pi}}. \tag{8}$$

The pseudo-incompressible approximation is valid when the Mach number is small and $\pi' \ll \overline{\pi}$ [5].

Yet another way to eliminate all vertically propagating sound waves is through the quasi-hydrostatic approximation, in which the vertical component of the momentum equation is replaced by the hydrostatic equation. Like (5) and (8), the time derivative is eliminated from the mass conservation relation for quasi-hydrostatic flow when the vertical coordinate is transformed from physical height z to some function of pressure $\zeta(p)$. The simplest such example is when $\zeta = p$ and the continuity equation becomes [8, p. 59]

$$\nabla_\zeta \cdot \mathbf{u_h} + \frac{\partial\dot{\zeta}}{\partial\zeta} = 0, \tag{9}$$

where ∇_ζ is the horizontal gradient operator along surfaces of constant ζ, $\dot{\zeta} = d\zeta/dt$, and $\mathbf{u_h}$ is the horizontal velocity vector. A second example due to Hoskins and Bretherton [9] uses the pseudo-height coordinate

$$\zeta = \frac{c_p\Theta}{g}\left[1 - \left(\frac{p}{p_s}\right)^{R/c_p}\right], \tag{10}$$

in which ζ is identical to the physical height z in an isentropic atmosphere with potential temperature Θ. The mass continuity equation for quasi-hydrostatic motions in pseudo-height coordinates may be expressed as

$$\nabla_\zeta \cdot (\rho\theta\mathbf{u_h}) + \frac{\partial\rho\theta\dot{\zeta}}{\partial\zeta} = 0. \tag{11}$$

Clearly this relation is similar to (8), but it is most accurate in problems where the characteristic horizontal scale greatly exceeds the vertical scale, whereas (8) becomes increasingly accurate as the horizontal and vertical scales approach each other, i.e., as the flow becomes more nonhydrostatic [13].

The main thrust of this paper is to extend the pseudo-incompressible approach to develop a unified framework applicable to low-Mach number flow in both quasi-hydrostatic and nonhydrostatic motions. We begin by further examining the parallels between the Boussinesq approximation for liquids and gases in section 2. The energy conservation properties of equations based on (5) and (6) are compared in section 3. Generalizations of the pseudo-incompressible equation are presented in section 4, along with a comparison to continuity equations for quasi-hydrostatic flow in pseudo-height coordinates. Section 5 contains the conclusions.

2. Comparing the Boussinesq approximation for liquids and gases

For inviscid flows under the action of a gravitational restoring force, the unapproximated momentum equation is

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p = -g\rho\mathbf{k}, \tag{12}$$

where p is pressure, g is the gravitational acceleration, and $\mathbf{k}$ is a unit vector along the vertical coordinate z. Let $\overline{\rho}(z)$ and $\overline{p}(z)$ characterize a vertically varying reference state in hydrostatic balance, such that

$$\rho(x, y, z, t) = \overline{\rho}(z) + \rho'(x, y, z, t),$$

$$p(x, y, z, t) = \overline{p}(z) + p'(x, y, z, t),$$

and

$$\frac{d\overline{p}}{dz} = -\overline{\rho}g. \tag{13}$$

Subtracting (13) from (12) yields an alternative form for the *unapproximated* momentum equations

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p' = -g\rho'\mathbf{k}. \tag{14}$$

Let ρ_0 be a constant reference density and define the pressure potential P, buoyancy b and Brunt-Väisälä frequency N such that

$$P = \frac{p'}{\rho_0}, \qquad b = -g\frac{\rho'}{\rho_0}, \qquad \text{and} \qquad N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz}; \tag{15}$$

then the conventional Boussinesq system for adiabatic inviscid flow can be expressed in the compact form

$$\frac{D\mathbf{u}}{Dt} + \nabla P = b\mathbf{k}, \tag{16}$$

$$\frac{Db}{Dt} + N^2 w = 0, \tag{17}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{18}$$

If the fluid in question is a liquid, the use of (17) is justified because it is equivalent to the incompressiblity condition (1), which as noted previously, also reduces the full mass continuity equation to (18). The remaining approximation in the application of the Boussinesq equations to liquids appears in the momentum equations, where the density in the first term in (14) is replaced by ρ_0, which amounts to neglecting the influence of density variations on momentum while retaining their influence on buoyancy [4].

Now consider the situation when the fluid is a compressible perfect gas and the flow is isentropic, as in many theoretical atmospheric applications. In this case analogous definitions for P, b and N^2 can be formulated in terms of π and θ. Using $(1/\rho)\nabla p = c_p\theta\nabla\pi$, taking $\overline{\theta}(z)$ and $\overline{\pi}(z)$ to be a vertically varying reference state in hydrostatic balance,

$$c_p\overline{\theta}\frac{d\overline{\pi}}{dz} = -g, \tag{19}$$

and letting primes denote deviations from this reference state, the unapproximated momentum equations for inviscid flow may be expressed in the alternative form

$$\frac{D\mathbf{u}}{Dt} + c_p\theta\nabla\pi' = g\frac{\theta'}{\overline{\theta}}\mathbf{k}. \tag{20}$$

The same Boussinesq equations (16)–(18) can serve as approximate governing equations for a perfect gas if

$$P = c_p\theta_0\pi', \qquad b = g\frac{\theta'}{\theta_0}, \qquad \text{and} \qquad N^2 = \frac{g}{\theta_0}\frac{d\overline{\theta}}{dz}, \tag{21}$$

where θ_0 is a constant potential temperature representative of the flow. Using these definitions, (17) is identical to the thermodynamic equation for isentropic flow

$$\frac{D\theta}{Dt} = 0. \tag{22}$$

The approximation required to obtain (16) from (20) is analogous to that used in the incompressible case: variations in potential temperature are neglected except in the leading-order contribution to the buoyancy.

The approximations required to arrive at (16) and (17) are therefore similar in compressible and incompressible fluids. On the other hand, the requirement for nondivergent flow is easily justified only for liquids, and the errors incurred approximating the true mass conservation relation by (18) can be quite large in stratified compressible flow. Let us therefore consider relations such as (5) and (6) that better approximate the true mass continuity equation in low-Mach number stratified compressible flow. The generalization of (5) to liquids is nevertheless of considerable interest, and is discussed by Bois [3].

3. Energy equations for the anelastic and pseudo-incompressible systems

One significant difference between the approximate mass continuity equations (5) and (6) is their impact on the energy conservation properties of the governing equations. This is most easily illustrated by deriving simplified energy equations in which it is assumed that the deviation A' of any thermodynamic variable A from its value in a horizontally uniform hydrostatically balanced reference state $\overline{A}(z)$ is small.

Linearizing the thermodynamic variables about their reference-state values, the momentum equation (20) may be approximated as

$$\frac{D\mathbf{u}}{Dt} = -\nabla(c_p\overline{\theta}\pi') + c_p\pi'\nabla\overline{\theta} + g\frac{\theta'}{\overline{\theta}}\mathbf{k}, \tag{23}$$

or using $c_p\overline{\theta}\pi' = p'/\overline{\rho}$,

$$\frac{D\mathbf{u}}{Dt} = -\nabla\left(\frac{p'}{\overline{\rho}}\right) + \left[\frac{d\ln\overline{\theta}}{dz}\frac{p'}{\overline{\rho}} + g\frac{\theta'}{\overline{\theta}}\right]\mathbf{k}. \tag{24}$$

An equation for the evolution of the kinetic energy, obtained by taking the dot product of $\overline{\rho}\mathbf{u}$ with (24), is

$$\overline{\rho}\frac{D}{Dt}\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) = -\nabla\cdot(p'\mathbf{u}) + \frac{p'}{\overline{\rho}}\left[\nabla\cdot(\overline{\rho}\mathbf{u}) + \overline{\rho}w\frac{d\ln\overline{\theta}}{dz}\right] + \overline{\rho}gw\frac{\theta'}{\overline{\theta}}. \tag{25}$$

The first term on the right side is the convergence of energy flux due to the work done by the pressure, the other two terms represent conversions between kinetic and other forms of energy. First consider the last conversion term. Multiply (19) by w and express the result as

$$\overline{\theta}\frac{Dc_p\overline{\pi}}{Dt} + \frac{Dgz}{Dt} = 0. \tag{26}$$

Linearizing the thermodynamic equation (7) as

$$\frac{D}{Dt}(\overline{\theta} + \theta') = \frac{\overline{\theta}Q}{c_p\overline{T}}, \tag{27}$$

multiplying this equation by $c_p\overline{\pi}$ and adding the result to (26), one may show

$$\overline{\rho}\frac{D}{Dt}\left(c_p\overline{\pi}\theta + gz\right) = -\overline{\rho}gw\frac{\theta'}{\overline{\theta}} + \overline{\rho}Q. \tag{28}$$

Having assumed that all thermodynamic perturbations about the mean state are small, and recalling that $T = \pi\theta$, $c_p\overline{\pi}\theta$ may be expressed as

$$c_p\overline{\pi}\theta = c_pT\left(1 - \frac{\pi'}{\pi}\right) = c_vT + RT\left(1 - \frac{p'}{p}\right) = c_vT + \frac{\overline{p}}{\rho} \tag{29}$$

The last expression, $c_vT + \overline{p}/\rho$, represents the *anelastic enthalpy* per unit mass, which differs from the exact enthalpy in that the influence of the perturbation pressure on changes in specific volume are neglected. Following standard terminology in atmospheric science, the sum of the anelastic enthalpy and the geopotential will be referred to as the anelastic dry static energy. Clearly the last term on the right side of (25) represents the exchange between kinetic energy and the anelastic dry static energy.

The second term on the the right side of (24) represents the conversion between kinetic energy and elastic energy. An equation for the evolution of the elastic energy can be derived as follows. The thermodynamic equation (7) may be alternatively linearized with respect to the perturbation potential temperature as

$$\frac{D}{Dt}\left(\frac{\theta'}{\overline{\theta}}\right) = -w\frac{d\ln\overline{\theta}}{dz} + \frac{Q}{c_p\overline{T}}, \tag{30}$$

which differs from (27) by a term of $O(\theta'/\overline{\theta})$. Both (27) and (30) agree with the true thermodynamic equation to within the accuracy of our linearization hypothesis that $A'/\overline{A} \ll 1$ for the thermodynamic variables. The continuity equation (3) may be alternatively expressed as

$$\frac{D}{Dt}\left[\ln\overline{\rho} + \ln\left(1 + \frac{\rho'}{\overline{\rho}}\right)\right] + \nabla\cdot\mathbf{u} = 0, \tag{31}$$

which linearizes as

$$\frac{D}{Dt}\left(\frac{\rho'}{\overline{\rho}}\right) = -w\frac{d\ln\overline{\rho}}{dz} - \nabla\cdot\mathbf{u} = -\frac{1}{\overline{\rho}}\nabla\cdot(\overline{\rho}\mathbf{u}). \tag{32}$$

The equation of state for perfect gases, $p = \rho RT$ may be alternatively expressed as

$$p^{1/\gamma} = Rp_s^{-R/c_p}\rho\theta, \tag{33}$$

where $\gamma = c_p/c_v$. Using (33),

$$\frac{1}{\gamma}\frac{p'}{\overline{p}} = \frac{\rho'}{\overline{\rho}} + \frac{\theta'}{\overline{\theta}}. \tag{34}$$

Taking D/Dt of (34), using (30) and (32), and treating the reference-state speed of sound $c_s = \gamma R\overline{T} = \gamma(\overline{p}/\overline{\rho})$ as a constant yields

$$\frac{1}{c_s^2}\frac{D}{Dt}\left(\frac{p'}{\overline{\rho}}\right) = -\frac{1}{\overline{\rho}}\left[\nabla\cdot(\overline{\rho}\mathbf{u}) + \overline{\rho}w\frac{d\ln\overline{\theta}}{dz}\right] + \frac{Q}{c_p\overline{T}}. \tag{35}$$

Multiplying (35) by p' one obtains

$$\overline{\rho}\frac{D}{Dt}\left[\frac{1}{2c_s^2}\left(\frac{p'}{\overline{\rho}}\right)^2\right] = -\frac{p'}{\overline{\rho}}\left[\nabla\cdot(\overline{\rho}\mathbf{u}) + \overline{\rho}w\frac{d\ln\overline{\theta}}{dz}\right] + \frac{p'Q}{c_p\overline{T}}, \tag{36}$$

which is the evolution equation for elastic energy per unit mass $(p'/\overline{\rho})^2/(2c_s^2)$ [12, p. 223]. Comparing (25) and (36), we see the second term on the right side of (25) represents the conversion from elastic to kinetic energy.

One of the most fundamental aspects of anelastic flow is that the elastic energy is either negligible or in equilibrium, and according to (36) this will be the case when

$$\nabla\cdot(\overline{\rho}\mathbf{u}) + \overline{\rho}w\frac{d\ln\overline{\theta}}{dz} = \frac{\overline{\rho}Q}{c_p\overline{T}}. \tag{37}$$

For adiabatic processes, the preceding reduces to

$$\nabla\cdot(\overline{\rho}\mathbf{u}) + \overline{\rho}w\frac{d\ln\overline{\theta}}{dz} = 0. \tag{38}$$

Most anelastic systems approximate the continuity equation as

$$\nabla\cdot(\overline{\rho}\mathbf{u}) = 0. \tag{39}$$

In the original anelastic system presented by Ogura and Phillips [14], the reference state is isentropic so that $\overline{\theta}$ is constant, in which case (39) is sufficient to guarantee satisfaction of (38). When (38) holds, the right side of the sum of (25) and (28) reduces to $-\nabla\cdot(p'\mathbf{u})$, so the total kinetic and anelastic dry static energy is conserved if the flow is isentropic and no work is done on the parcel by perturbation pressure forces.

The magnitude of the potential temperature perturbations about an isentropic reference state are, however, often much larger than those that could be obtained using a vertically varying $\overline{\theta}(z)$ that more closely matches the mean atmospheric structure. To reduce the size of the thermodynamic perturbations with respect to stably stratified reference states, Wilhelmson and Ogura [16] allowed $\overline{\theta}$ to vary in the vertical, but since they used the continuity equation (39), the resulting system was not energy conservative.

An energy conserving system that uses (39) and allows vertical variations in $\overline{\theta}$ was, however, obtained by Lipps and Hemler [11] by modifying the momentum equations. They neglected the second term on the right side of (23) which yields the approximate momentum equation

$$\frac{D\mathbf{u}}{Dt} = -\nabla(c_p\overline{\theta}\pi') + g\frac{\theta'}{\overline{\theta}}\mathbf{k}, \tag{40}$$

and also eliminates the second term in the brackets on the right side of (25), thereby ensuring that satisfaction of (39) will make the system energy conservative.

A functionally equivalent approximation was made by Bannon [1], but rather than neglecting the second term on the right side of (23), he defined "dynamic entropy" θ_d' such that

$$\frac{\theta_d'}{\overline{\theta}_d} = -\frac{p'}{\overline{\rho}g}\frac{d\ln\overline{\rho}}{dz} - \frac{\rho'}{\overline{\rho}}. \tag{41}$$

The true perturbation potential temperature satisfies (34), and is equal to θ'_d only when the reference state is isentropic. Using (34), (41) and

$$\frac{d\ln\overline{\theta}}{dz} = \frac{1}{\gamma}\frac{d\ln\overline{p}}{dz} - \frac{d\ln\overline{\rho}}{dz} = -\frac{\overline{\rho}g}{\gamma\overline{p}} - \frac{d\ln\overline{\rho}}{dz}, \tag{42}$$

the second term on the right side of (24) may be written as $g\theta'_d\mathbf{k}/\overline{\theta}_d$. He closed the system consisting of (24) and (39) with an approximate thermodynamic equation in which θ' in (27) is replaced by θ'_d. The set of equations for $(\mathbf{u}, c_p\overline{\theta}\pi', \theta')$ in the Lipps-Hemler system are identical to the set for $(\mathbf{u}, p'/\overline{\rho}, \theta'_d)$ in Bannon's system, so they both have the same energy conservation properties.

Another approach that also allows arbitrary vertical variations in $\overline{\theta}(z)$ while still ensuring energy conservation is to simply enforce (37), which is equivalent to using (6) as the approximate continuity equation [5]. In the next section we consider generalizations of this approach.

The preceding energy relations are derived under the assumption that the perturbations of the thermodynamic variables about their reference state are small. Exact energy conservation relations that hold for finite-amplitude perturbations have also been derived for most of these anelastic systems. These energy equations express the local rate of change of some approximation to the total energy in an Eulerian volume as the energy flux divergence through the boundaries of that volume. In the anelastic equations of Lipps and Hemler [11], for example, (27), (39), and (40) form such an energy conservative set. Energy conservation is achieved in the pseudo-incompressible system of Durran [5] using (37) and the *unapproximated* thermodynamic and momentum equations, (7) and (20).

4. Generalized Pseudo-Incompressible Equations

Durran [7] noted that the pseudo-density used in the approximation to the mass continuity equation (6) may be generalized to allow horizontal variations in the hydrostatically balanced reference state such that $\rho^* = \overline{\rho}(x,y,z)\overline{\theta}(x,y,z)/\theta$. Allowing such variations is useful, for example, in global atmospheric modeling because there are significant equator-to-pole variations in the zonal-mean thermodynamic fields, and the magnitude of the perturbations about the reference state can be reduced if the reference-state values are set equal to the zonal mean. The continuity equation arising from this generalized form for ρ^*, together with the unapproximated momentum and thermodynamic equations, filter sound waves and have attractive energy-conservation properties. Other generalizations are also possible, and in the following we pursue a slightly different approach.

Consider approximations to the prognostic equation for pressure obtained by taking the convective derivative of the equation of state (33),

$$\frac{Dp^{1/\gamma}}{Dt} = Rp_s^{-R/c_p}\left(\theta\frac{D\rho}{Dt} + \rho\frac{D\theta}{Dt}\right), \tag{43}$$

or

$$p^{-R/c_p}\frac{\partial p}{\partial t} + Rp_s^{-R/c_p}\mathbf{u}\cdot\nabla(\rho\theta) = Rp_s^{-R/c_p}\left(-\rho\theta\nabla\cdot\mathbf{u} + \frac{\rho\theta Q}{c_pT}\right), \tag{44}$$

or finally

$$\frac{\theta}{c_s^2}\frac{\partial p}{\partial t} + \nabla\cdot(\rho\theta\mathbf{u}) = \frac{\rho Q}{c_p\pi}. \tag{45}$$

Sound waves will be eliminated from the governing equations if the first term is neglected to yield

$$\nabla \cdot (\rho\theta\mathbf{u}) = \frac{\rho Q}{c_p \pi}. \tag{46}$$

The preceding is a generalization of the pseudo-incompressible equation (8) and reduces to that equation if it is linearized under the assumption that the perturbation thermodynamic fields are small compared to a horizontally uniform hydrostatically balanced reference state. As a consequence, (46) retains the favorable energy conservation properties derived in section 3. Nevertheless, in contrast to the generalized pseudo-incompressible system in [7], Eulerian energy conservation relations for finite-amplitude thermodynamic perturbations have not been derived for the system consisting of (46) and the unapproximated momentum and thermodynamic equations.

Note that the first term in the *prognostic pressure equation* (45) will be zero in a coordinate system in which the pressure replaces physical height as the vertical coordinate. In fact *for quasi-hydrostatic flow*, close relatives to (46) and the pseudo-incompressible equation (8) can be obtained by simply transforming *the mass continuity equation* from physical height coordinates (x, y, z, t) to a system with z replaced by an appropriately defined vertical coordinate ζ. Under such a transformation the continuity equation becomes

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial \zeta}\right) + \nabla_\zeta \cdot \left(\rho\mathbf{u_h}\frac{\partial z}{\partial \zeta}\right) + \frac{\partial}{\partial \zeta}\left(\rho\dot{\zeta}\frac{\partial z}{\partial \zeta}\right) = 0, \tag{47}$$

where ∇_ζ is the horizontal gradient operator along surfaces of constant ζ, $\dot{\zeta} = d\zeta/dt$, and $\mathbf{u_h}$ is the horizontal velocity vector [6, p. 376].

Now suppose that ζ is "pseudo-height," defined by (10), or equivalently

$$\zeta = \frac{c_p\Theta}{g}(1 - \pi). \tag{48}$$

Then $\partial\pi/\partial\zeta = -c_p\Theta/g$, and from the hydrostatic relation

$$\left(\frac{\partial z}{\partial \pi}\right)_{x,y,t} = -\frac{c_p\theta}{g}, \tag{49}$$

thus

$$\frac{\partial z}{\partial \zeta} = \left(\frac{\partial z}{\partial \pi}\right)_{x,y,t}\frac{\partial \pi}{\partial \zeta} = \frac{\theta}{\Theta}. \tag{50}$$

Using (50) and recalling that Θ is a constant, in pseudo-height coordinates (47) becomes

$$\frac{\partial}{\partial t}(\rho\theta) + \nabla_\zeta \cdot (\rho\theta\mathbf{u_h}) + \frac{\partial}{\partial \zeta}\left(\rho\theta\dot{\zeta}\right) = 0. \tag{51}$$

The first term in the preceding is zero since the time derivative is taken at constant ζ, so (51) reduces to (11). The quasi-hydrostatic approximation eliminates all acoustic-wave solutions to the governing equations except the horizontally propagating Lamb wave [6, p. 375]. The Lamb wave can be eliminated by the specification of appropriate lower boundary conditions, such as $\dot{\zeta} = 0$.

Thus, like (46), (11) effectively filters the acoustic modes. Although they look similar, there are nevertheless several key differences between (46) and (11). For one thing, the operator in (11) is not simply the divergence in pressure coordinates, since that would require each velocity component to be multiplied by a factor of $\partial z/\partial\zeta$. More substantially, (46) is the prognostic pressure equation with the local time derivative neglected, and therefore includes a heating term. On the other hand, (11) is simply the continuity equation for quasi-hydrostatic flow in transformed coordinates, so it does not contain a term related to heating.

5. Conclusions

At least for perfect gases, the Boussinesq equations for the evolution of the momentum and buoyancy fields, (16) and (17), can be extended to compressible flows simply through an appropriate definition of the Boussinesq pressure potential, buoyancy and Brunt-Väisälä frequency. The approximation of the Boussinesq mass conservation relation (18) presents a greater challenge, and its generalization through the formulation of various anelastic continuity equations has been the focus of this paper.

Simple equations were derived for the changes in the kinetic, anelastic dry static, and elastic energies under the assumption that the perturbations to the thermodynamic variables were small compared to their values in a horizontally uniform, hydrostatically balanced reference state. Under these conditions, even when no work is done on the parcel by perturbation pressure forces, the traditional anelastic continuity equation (5) is not sufficient to ensure the conservation of the sum of kinetic and anelastic dry static energy unless additional approximations are made in the momentum equations or the thermodynamic equation. On the other hand, if the pseudo-density is defined as $\rho^* = \overline{\rho}\overline{\theta}/\theta$, the pseudo-incompressible continuity equation (6) combines with the thermodynamic equation to yield (8), which is precisely the condition required to conserve the total kinetic and anelastic dry static energy. Energy conservation is therefore achieved in the pseudo-incompressible system without making further approximations in the governing equations.

Sound waves continue to be filtered from the governing equations if the continuity equation is replaced by (46), which is both a generalization of the pseudo-incompressible equation (8) and a quasi-steady approximation to a prognostic equation for the pressure. The system consisting of (7), (20) and (46) is identical to the pseudo-incompressible system when the perturbations to the thermodynamic variables are small compared to their reference-state values, and it has the same favorable energy conservation properties in that limit.

Vertically propagating sound waves are also filtered by the quasi-hydrostatic approximation, and if the vertical coordinate is transformed to pseudo-height using (10), the result is (11), which is similar to the generalized pseudo-incompressible equation (46). The quasi-hydrostatic approximation is most accurate when the horizontal length scale greatly exceeds the vertical scale, whereas the accuracy of the generalized pseudo-incompressible approximtion increases as the horizontal and vertical scales become more comparable. In multi-scale modeling it would be advantageous to arrive at a single approximation that combines the merits of both (11) and (46).

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