ATMS442: Governing Equations

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla P - f\,\hat{k} \times \vec{V} \tag{1}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g \tag{2}$$

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \vec{U} \tag{3}$$

$$\frac{D\theta}{Dt} = 0 \tag{4}$$

Notes

- Assumptions: neglected sphericity terms, friction, heating, and conduction.
- $\vec{U} = (u, v, w)$ = velocity vector (m/s). $\vec{V} \equiv (u, v)$.
- P = pressure (Pa); ρ = density (kg/m^3); $P = \rho\, R\, T$.
- $\theta = T\left(\frac{P_{00}}{P}\right)^{R/C_p}$ = potential temperature (K); T = temperature (K).
- g = effective gravity (m/s^2).
- $f = 2\,\Omega \sin\phi$ = Coriolis parameter (s^{-1}); ϕ = latitude.
- $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{U} \cdot \nabla = \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + w\,\frac{\partial}{\partial z}$
  Lagrangian change = local change + advection.

Important force balances:

(1) $\frac{D\vec{V}}{Dt} \approx 0 \Longrightarrow \vec{V}_g = \frac{1}{\rho f}\,\hat{k} \times \nabla P$ geostrophic balance.

(2) $\frac{Dw}{Dt} \approx 0 \Longrightarrow \frac{\partial P}{\partial z} = -\rho\, z$ hydrostatic balance.

ATMS442: Boussinesq Equations

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho_0}\nabla P - f\,\hat{k} \times \vec{V} \tag{5}$$

$$\rho_0 \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} - \rho g \tag{6}$$

$$\nabla \cdot \vec{U} = 0 \tag{7}$$

$$\frac{D\theta}{Dt} = 0 \tag{8}$$

Notes

• Assumptions: Constant density (ρ_0) everywhere except in the bouyancy term.

$\Longrightarrow$ This assumes that the time changes of density are small, which is fine for shallow motions, but not for deep motions (e.g. convection).

ATMS442: Shallow Water Equations

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho_0}\nabla P - f\,\hat{k} \times \vec{V} \tag{9}$$

$$\frac{\partial P}{\partial z} = -\rho_0\, g \tag{10}$$

$$\nabla \cdot \vec{U} = 0 \tag{11}$$

$$\frac{D\theta}{Dt} = 0 \tag{12}$$

Notes

• Assumptions: Constant density (ρ_0), completely incompressible fluid with depth $h(x, y)$; hydrostatic balance.

• $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla = \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y}$

• Integrating the hydrostatic equation gives $P(z) - P(0) = -\rho_0 g z + f(x, y)$. The pressure at $z = h$ is a constant (C), which implies $f = \rho_0 g h - P(0) + C$, and:

$$P(z) = \rho_0 g(h - z) + C \tag{13}$$

• Since h is a function of (x, y) only, then so is P, and by (9) the tendency of $\vec{V}$ is a function of (x, y) too. Going one step further, if we assume that *initially* $\vec{V}$ doesn't depend on z, then it never will.

• Using the definition for θ, and constant density, one can show that (12) is the same as:

$$\frac{DP}{Dt} = 0. \tag{14}$$

• Applying (14) to (13) we find that $\dfrac{DP}{Dt} = \dfrac{Dh}{Dt} - \dfrac{Dz}{Dt} = 0$. Since $\dfrac{Dz}{Dt} = w$, (14) is the same as:

$$\frac{Dh}{Dt} = w. \tag{15}$$

• Finally, integrating the continuity equation over depth h, we find $w = -h\,(\nabla \cdot \vec{V})$, which eliminates w from (15):

$$\frac{Dh}{Dt} = -h\,(\nabla \cdot \vec{V}). \tag{16}$$

• Our final form of the shallow water equations uses (13) to express the pressure gradient term in (9) in terms of h.

Shallow Water Equations

$$\frac{D\vec{V}}{Dt} = -g\nabla h - f\,\hat{k} \times \vec{V} \tag{17}$$

$$\frac{Dh}{Dt} = -h\nabla \cdot \vec{V}. \tag{18}$$

More notes:

• If $\nabla \cdot \vec{V} = 0$, the shallow water equations simplify to a single equation, (17). One can show that this is the same as the **barotropic vorticity equation**:

$$\frac{D}{Dt}(\zeta + f) = 0 \tag{19}$$

where $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$.

ATMS442: Quasigeostrophic Equations

$$\frac{D_g \vec{V}_g}{Dt} = -\nabla\Phi - f\,\hat{k} \times \vec{V} \tag{20}$$

$$\frac{\partial \Phi}{\partial P} = -\frac{RT}{P} \tag{21}$$

$$\nabla \cdot \vec{V}_a = \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{22}$$

$$\frac{D_g \theta}{Dt} + \omega \frac{d\theta_0}{dP} = 0 \tag{23}$$

Notes

- Assumptions: Pressure as vertical coordinates (decreases upward!); a reference atmosphere that depends on P only (subscript "0"); "f" plane.
- $\omega = \dfrac{DP}{Dt}$ is the vertical motion; it is negative for rising air.
- Φ is the geopotential (units: $\mathrm{m}^2\ \mathrm{s}^{-2}$).
- $\vec{V}_a = \vec{V} - V_g$ is the "ageostrophic" wind.
- $\nabla \cdot \vec{V}_g = 0$, the geostrophic wind has no divergence.
- $\dfrac{D_g}{Dt} = \dfrac{\partial}{\partial t} + \vec{V}_g \cdot \nabla = \dfrac{\partial}{\partial t} + u_g \dfrac{\partial}{\partial x} + v_g \dfrac{\partial}{\partial y}$

  Lagrangian change = local change + *geostrophic* advection.