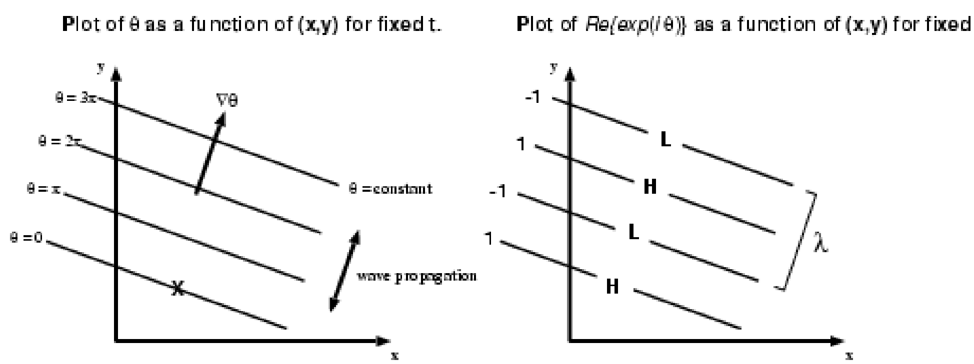
Plane-Wave Summary

A two-dimensional plane wave may be expressed as

$$f(x,y,t) \;=\; Re\left\{\, A\, e^{i(kx+ly-\nu t)} \right\} \;=\; Re\left\{\, A\, e^{i\theta} \right\} \tag{1}$$

- x, y and t are independent variables (space and time).
- k and l are the x and y *wavenumbers* (units: m^{-1}).
- ν is called the wave *frequency*.
- A is the wave *amplitude*.
- by convention, k and l may be positive or negative, and ν is positive.
- $\theta = kx + ly - \nu t$ is the wave *phase angle*.
- The wave *propagates* normal to lines of constant phase angle.

At any instant in time [t fixed; (x, y) varies]:

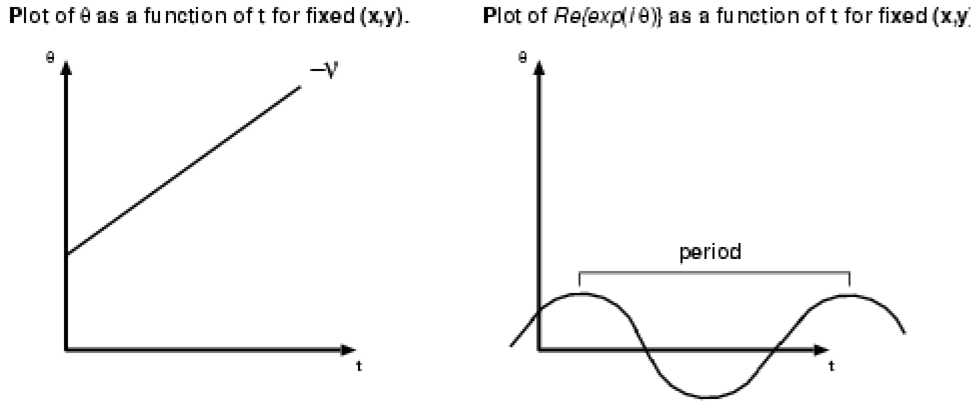


- $\theta = kx + ly + C$; θ is a linear function of space.
- θ is constant on lines of $kx + ly$. This implies $d\theta = \frac{\partial\theta}{\partial x}\delta x + \frac{\partial\theta}{\partial y}\delta y = 0$, for θ constant.

Therefore, the slope of these lines is $\frac{\delta y}{\delta x}|_\theta = -k/l$.

- $e^{i\theta} = e^{i(\theta + 2\pi n)}$, where n is an integer, are lines of *constant phase* (e.g. highs and lows).
- $\vec{K} = \nabla\theta = \hat{\imath}k + \hat{\jmath}l$ is the *wave vector*; $\mathcal{K} = |\vec{K}|$ is the *wavenumber*.
- $\lambda = \frac{2\pi}{\mathcal{K}}$ is the *wavelength*: the distance between lines of constant phase.

At any fixed point in space $[(x, y)$ fixed; t varies]:



- $\theta = C - \nu t$; θ is a linear function of time.
- $\nu = -\dfrac{\partial \theta}{\partial t}$, is the *frequency*: the rate that lines of constant phase pass a fixed point in space (units: s^{-1}). Note that the figure above indicates $\nu < 0$. This means that for fixed (x, y), such as the point marked "**X**" on the first figure, θ increases with time; this can only occur if phase lines move toward smaller x and y.
- The wave *period* is $\dfrac{2\pi}{\nu}$: length of time between points of constant phase (units: s).
- The *phase speed* is the propagation speed of constant phase lines in the direction of $\vec{K}$, $c = \dfrac{\nu}{\mathcal{K}} = -\dfrac{1}{|\nabla \theta|}\dfrac{\partial \theta}{\partial t}$ (units: m s^{-1}).

Special note on θ:

If θ has an *imaginary part*, $\theta = \theta_r + i\theta_i$, then $e^{i\theta} = e^{i(\theta_r + i\theta_i)} = e^{i\theta_r}\, e^{-\theta_i} \equiv A^* e^{i\theta_r}$. θ_r is the wave phase angle as interpreted above, and $A^* = Ae^{-\theta_i}$ is a modified amplitude that depends on time and/or space. For example, if the frequency, ν, contributes the imaginary part, then the wave has *time-dependent* amplitude that grows or decays with time. Such waves are called *unstable*, to distinguish them from the *neutral* waves (A constant) that we discussed above.