

ATMS542 Fluid Mechanics Review

Governing Equations

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla_H P - f\hat{k} \times \vec{V} \quad (1)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g \quad (2)$$

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \vec{U} \quad (3)$$

$$\frac{D\theta}{Dt} = 0 \quad (4)$$

Notes

- Assumptions: neglected sphericity terms, friction, heating, and conduction.
- $\vec{U} = (u, v, w)$ = velocity vector (m/s). $\vec{V} \equiv (u, v)$.
- P = pressure (Pa); ρ = density (kg/m³); T = temperature (K).
- $P = \rho R T$; $R = 287$ J/(K kg); technically $T = T_v$, the virtual temperature.
- $\theta = T \left(\frac{P_{00}}{P} \right)^{R/C_p}$ = potential temperature (K).
- g = effective gravity (m/s²).
- $f = 2\Omega \sin \phi$ = Coriolis parameter (s⁻¹); ϕ = latitude.
- $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{U} \cdot \nabla = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$
Lagrangian change = local change + advection.

Summary: 6 equations and 6 unknowns (u, v, w, P, ρ, T)

Ertel potential-vorticity conservation law (incorporates all previous laws):

$$\frac{D\Pi}{Dt} = 0. \quad (\text{adiabatic and frictionless}) \quad (5)$$

where

$$\Pi = \frac{\vec{\omega} + 2\vec{\Omega}}{\rho} \cdot \nabla\theta. \quad (\text{Ertel PV}) \quad (6)$$

$$\vec{\omega} = \nabla \times \vec{U} = (\xi, \eta, \zeta) \quad (\text{vorticity vector}) \quad (7)$$

Important Parameters and Dominant Force Balances

Horizontal length scale: $(x, y) \sim L \sim 1000$ km.

Vertical length scale: $z \sim H \sim 10$ km.

Aspect ratio: $\delta = H/L \sim 0.01$.

Horizontal velocity scale: $(u, v) \sim U \sim 10$ m/s. Since $U \sim L/T$, this implies ...

Time scale: $t \sim T = L/U \sim 28$ h.

Vertical velocity scale: $w \sim H/T = \frac{HU}{L} = \delta U \sim 10$ cm/s.

Ratio of horizontal accelerations to Coriolis effect (Rossby number):

$$\frac{\rho \frac{DV}{Dt}}{\rho f \hat{k} \times \vec{V}} \sim \frac{U}{fL} \equiv \epsilon \sim 0.1 \quad (8)$$

Geostrophic balance in horizontal ($\epsilon \ll 1$):

$$0 \sim -\nabla_H P - \rho f \hat{k} \times \vec{V} \implies \vec{V} \approx \frac{1}{\rho f} \hat{k} \times \nabla_H P \quad (9)$$

Hydrostatic balance in vertical ($\delta \ll 1$):

$$\frac{Dw}{Dt} \sim \epsilon \delta \ll 1 \implies \frac{\partial p}{\partial z} \approx -\rho g \quad (10)$$

Note: (10) and (9) mean that accelerations are small relative to other terms.

Constant-Density Limiting Fluids

(1) Shallow Water Equations

Assumptions: $\rho = \text{constant}$; $\delta \ll 1$; adiabatic.

$$\rho \frac{D\vec{V}}{Dt} = -\nabla_H P - \rho f \hat{k} \times \vec{V}. \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla_H \quad (11)$$

$$\nabla_H \cdot \vec{V} = -\frac{1}{h} \frac{Dh}{Dt} \quad (h = \text{fluid depth}) \quad (12)$$

$$\frac{\partial p}{\partial z} = -\rho g \quad (13)$$

PV conservation (5)–(7) becomes

$$\frac{D}{Dt} \left[\frac{\zeta + f}{h} \right] = 0. \quad (14)$$

Note: because the fluid supports both *vorticity* and *divergence*, the velocity field may be represented by a nondivergent *streamfunction* plus an irrotational *velocity potential* (Helmholtz Theorem):

$$\vec{u} = \nabla \times \vec{\Psi} + \nabla \chi \quad (15)$$

$$\nabla \times \vec{u} = \nabla \times \nabla \times \vec{\Psi} \quad \nabla \cdot \vec{u} = \nabla^2 \chi \quad (16)$$

(2) Barotropic Vorticity Equation

Assumptions: Shallow water *and* $h = \text{constant}$.

PV conservation (14) becomes

$$\frac{D}{Dt}(\zeta + f) = 0. \quad (17)$$

Note: because this fluid supports only *vorticity*, and is strictly two dimensional [$\zeta = f(x, y)$],

(15) allows the velocity field to be represented by a scalar nondivergent *streamfunction*:

$$\vec{V} = \nabla \times \hat{k}\psi = \hat{k} \times \nabla_H \psi \quad \zeta = \nabla_H \times \vec{V} = \nabla_H^2 \psi \quad (18)$$

Using the β -plane approximation, $v \frac{\partial f}{\partial y} \approx v\beta$, and (17) becomes

$$\frac{D}{Dt} \zeta + v\beta = 0. \quad (19)$$

(19) has plane-wave solutions of the form

$$\psi(x, t) = A e^{i(kx - \sigma t)} \quad (20)$$

provided

$$\sigma = -\frac{\beta}{k} \quad (\text{Rossby waves}). \quad (21)$$