What is the thermodynamic chart?

Begin with the 1st law of thermodynamics:

$$dq \equiv TdS = c_v dT + dw \qquad [1]$$

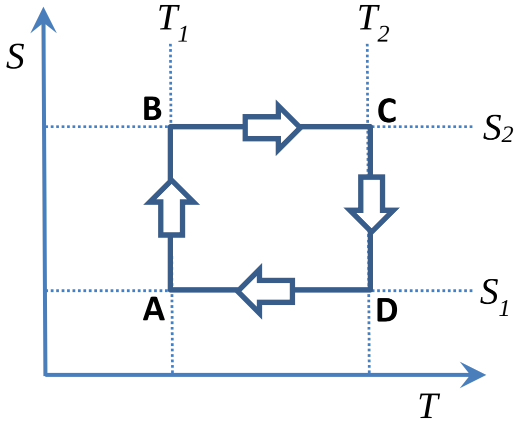
Where dq is the heat supplied $\equiv TdS$ where T is temperature and dS is the entropy change, c_v is the heat capacity of air at constant volume, dT is the temperature change, and dw is the work done, which is itself equal to $p d\alpha$ where p is the pressure and $d\alpha$ is the change in specific volume.

With the product rule trick, $d(p\alpha) = p d\alpha + \alpha dp$, and noting that $p\alpha = RT$ from the ideal gas law (R being the gas constant for air) we can substitute into [1] to obtain

$$dq \equiv TdS = c_v dT + RdT - \alpha dp = c_p dT - \alpha dp \qquad [2]$$

or, simply: $$TdS = c_p dT - \alpha dp \qquad [3]$$

Consider a coordinate system with S and T as the orthogonal axes:



For processes in which S is constant, Eqn. [1] shows that $c_v dT + dw = 0$, so that the work done going from **B** to **C** is canceled by that going from **D** to **A**.

For processes in which T is constant, Eqn. [1] shows that $TdS = dw$, so that the work done going from **A** to **B** is $T_1(S_2 - S_1)$ whereas the work done from **C** to **D** is $T_2(S_1 - S_2)$.

The net work done in traversing the circuit **A→B→C→D→A** is $T_2(S_1 - S_2) + T_1(S_2 - S_1) = (T_1 - T_2)(S_2 - S_1)$, which is the area enclosed by the circuit **ABCD** on the chart.

This is important, because it allows us to determine the amount of work done by a gas graphically.

We can integrate Eqn. [3], recalling that $\frac{\alpha}{T} = \frac{R}{p}$ (ideal gas law), to give:

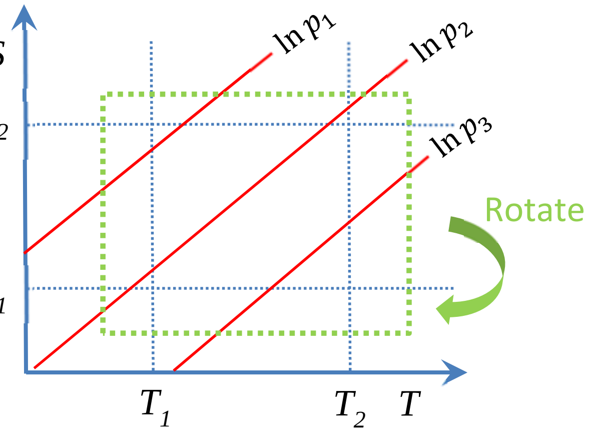
$$S = S_0 + c_p \ln\frac{T}{T_0} - R\ln\frac{p}{p_0} \qquad [4]$$

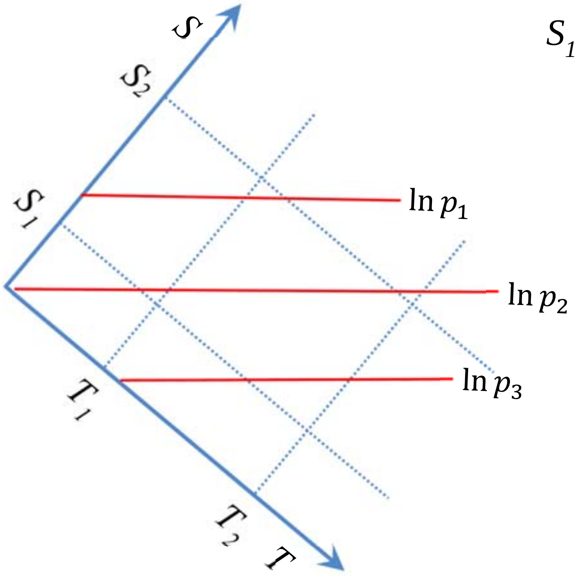
Where the subscripts “0” are constants of integration. Alternatively, [4] can be written as

$$\ln p = \text{const.} + \frac{1}{k}\ln\frac{T}{T_0} - \frac{S}{R} \tag{5}$$

where $k = \frac{R}{c_p}$.

From [5] it can be seen that $\ln p$ is a family of curves in *S*-*T* coordinates [right]





This can be rotated such that the $\ln p$ isolines are horizontal [left]……

……which is beginning to resemble the skew-T chart.

Lines of constant *S* can be labeled lines of constant potential temperature θ because, from the definition of potential temperature:

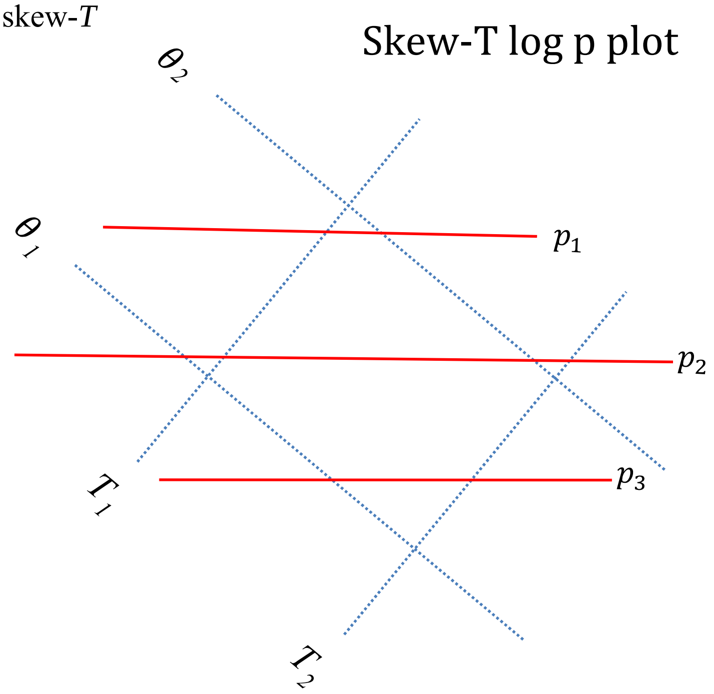
$$\theta = T\left(\frac{p_0}{p}\right)^k, \tag{6}$$

and taking logs of [6], we obtain

$$c_p \ln\theta = c_p \ln T - R\ln p + R\ln p_0 \tag{7}$$

Comparing with [4] shows that $S \propto \ln\theta$. This also reminds us that constant θ processes are isentropic ($dq = dS = 0$).

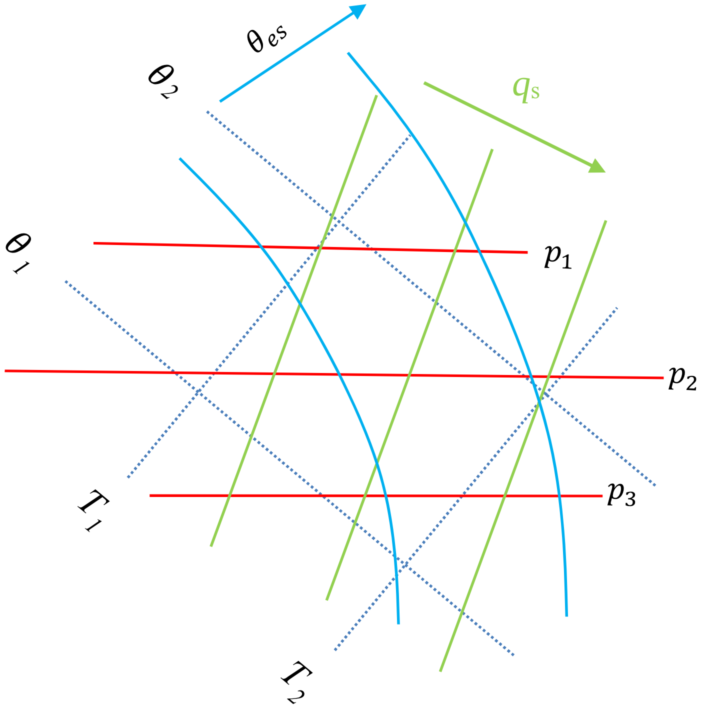
Note that the true coordinates of the skew-T log p plot are T and θ.



Skew-T log p plot

Some key properties of the chart:

- *Any* functions of T and p can be plotted on the chart, e.g. potential temperature, saturation mixing ratio ($q_s \approx \varepsilon e_s(T)/p$), saturation equivalent potential temperature (θ_{es}). This leads to additional families of curves representing constant values of the aforementioned functions.



Derivation of saturation equivalent potential temperature: From the 1st law [3], substituting the ideal gas law to eliminate α, we arrive at

$$dq = c_p dT - RT\frac{dp}{p} \quad [8]$$

When a parcel is saturated and condenses out all vapor above saturation, latent heat is released such that $dq = -L_v dq_s(T,p)$. Note the negative sign because $dq_s < 0$ as a parcel is ascending (and cooling). Substituting this into [8], and using the differential form for potential temperature [7] to eliminate p, we obtain:

$$-L_v dq_s(T,p) = c_p dT - c_p T\left(\frac{dT}{T} - \frac{d\theta}{\theta}\right) = c_p \frac{T}{\theta} d\theta \quad [9]$$

From which we can integrate to obtain an analytical form. However, we need to make an approximation, namely that $\frac{L_v dq_s(T,p)}{c_p T} \approx d\left(\frac{L_v q_s(T,p)}{c_p T}\right)$, which leads to

$$\int \frac{d\theta}{\theta} \approx -\int d\left(\frac{L_v q_s(T,p)}{c_p T}\right) \quad [10]$$

Typically, we integrate from the location on the chart (T, p, θ) to the top of the atmosphere (T=0, q_s=0) where all the moisture condenses out, s.t. and the potential temperature at that point is defined as the **saturated equivalent potential temperature** θ_{es}

$$[\ln\theta]_{\theta}^{\theta_{es}} \approx \frac{L_v q_s(T,p)}{c_p T} \quad [11]$$

From which one obtains

$$\theta_{es} \approx \theta \exp\frac{L_v q_s(T,p)}{c_p T} \quad [12]$$

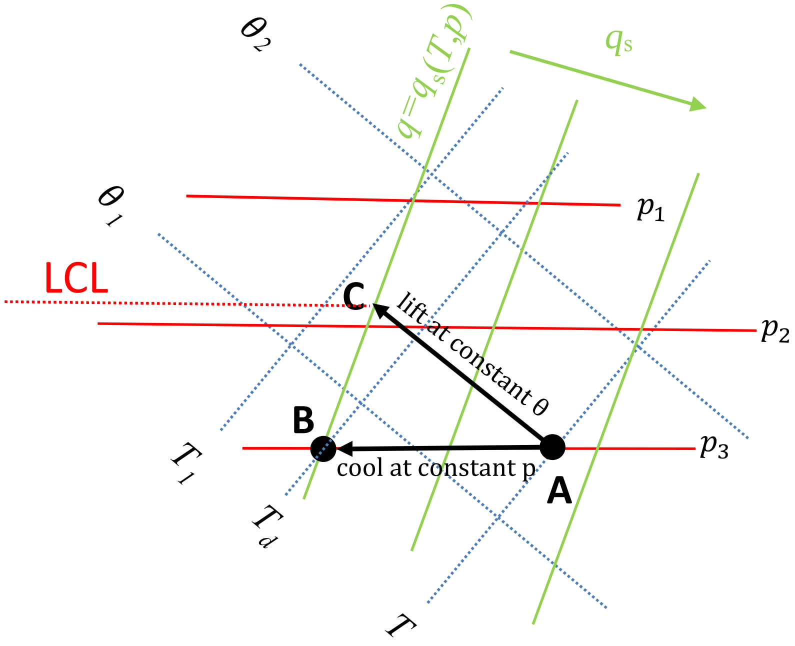
This is the most commonly used form of the saturation equivalent potential temperature. It is the potential temperature of a parcel that one would obtain assuming the parcel is saturated and then all its water vapor is condensed out. This variable is conserved under moist adiabatic processes.

Thermodynamic variables that depend upon T, p **and** q can also be shown. For example, the **dewpoint temperature** (T_d) is the temperature to which an air parcel must be cooled *at constant pressure* to obtain saturation.

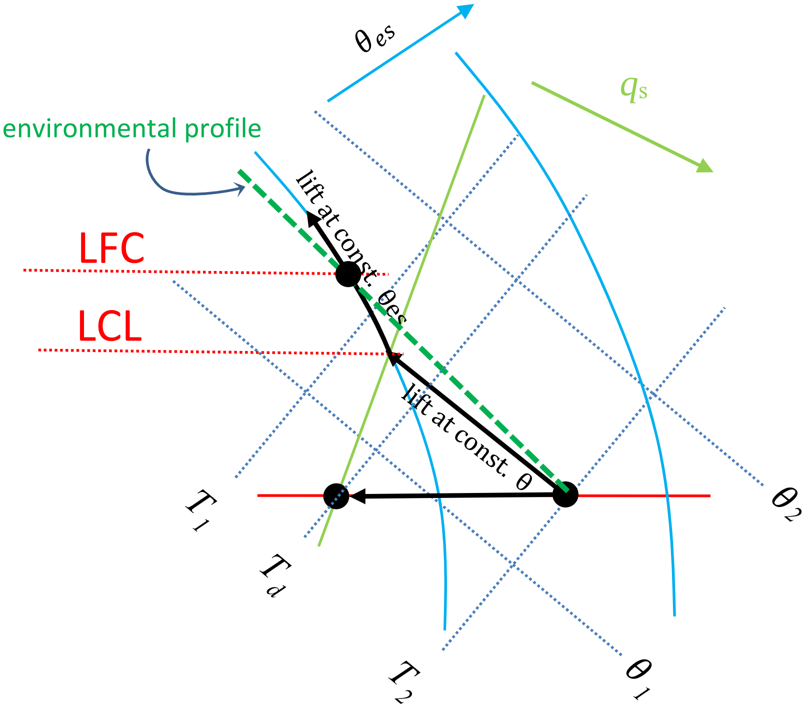
Knowing T_d and T allows us to find the vapor mixing ratio q from the saturation mixing ratio line intersecting the T_d, p point (point B below). Then, **relative humidity** (RH) can be found from $q/q_s(T,p)$

Lifting condensation level (LCL) can be found by identifying the parcel's mixing ratio using T and T_d as below, and then raising a parcel from point A (below) up the dry adiabat (constant θ) to the point where the parcel's mixing ratio equals the saturation mixing ratio (point C below).

Equivalent potential temperature can be determined in a similar fashion, since it requires both the temperature of the *LCL*, T and q.

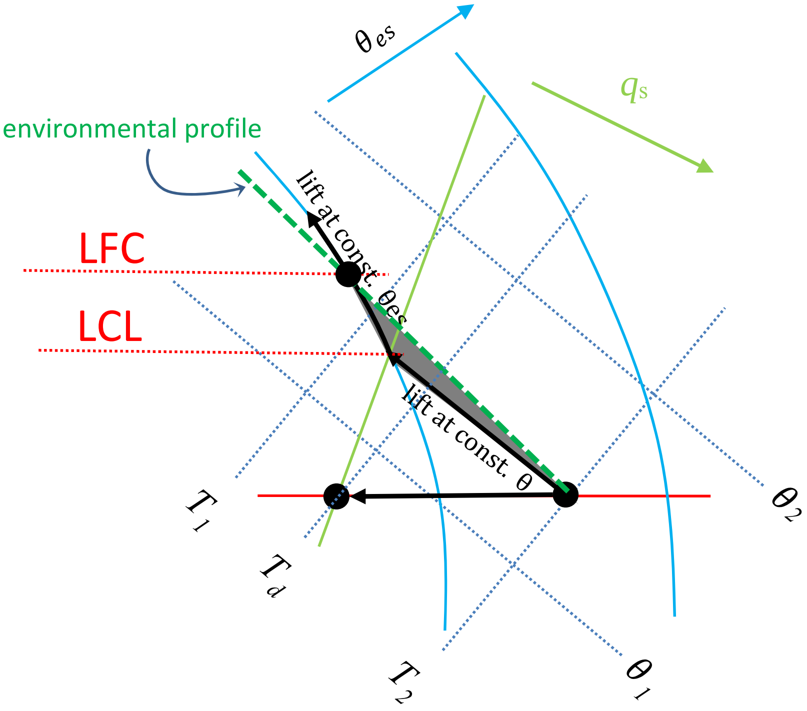


Level of free convection (LFC). This is the level at which a parcel lifted from the surface under absolutely stable conditions, becomes positively buoyant following saturation and further lifting. This is shown at right (top).



Convective inhibition (CIN) is the amount of energy required to overcome the lifting of a parcel from a level to another level, i.e. the amount of work required to overcome the negatively buoyant energy exerted by the environment on an air parcel. **Most commonly, CIN is calculated for lifting from the surface to the LFC.**

On a thermodynamic chart, the net work done is the area enclosed by the movement in [T,θ] space, which is equal to the gray shaded area (right).

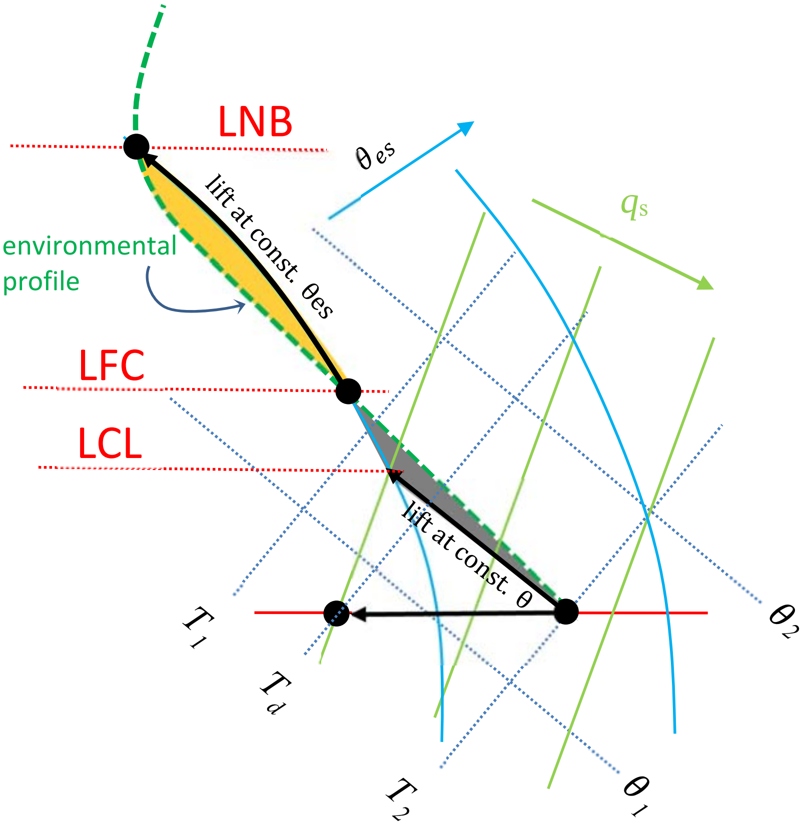


The CIN can therefore be estimated by the vertically-integrated buoyancy deficit of the ascending parcel compared with its environment:

$$CIN = \int_0^{LFC} g\left(\frac{T_{parcel}-T_{env}}{T_{env}}\right) dz \qquad [13]$$

High values of CIN imply that considerable work must be done to lift a parcel to where it becomes positively buoyant.

The **convectively available potential energy (CAPE)** is the work done on the parcel by the environment once it has reached the level of free convection (LFC) and continues to move upward with positive buoyancy until it reaches the level where it becomes neutrally buoyant again (the **level of neutral buoyancy LNB**). In a similar fashion to the CIN, CAPE is estimated using an integral of the buoyancy excess of the parcel compared with the environment:



$$CAPE = \int_{LFC}^{LNB} g \left(\frac{T_{parcel} - T_{env}}{T_{env}}\right) dz \qquad [14]$$

Note that conditional instability (positive CAPE) is only possible if the slope of the environmental profile is such that $\frac{d\theta_{es}}{dz} < 0$.