ATMS 532 Homework 1 answers January 2007

14 points total.

Petty 1.2 3 points. Atmospheric heating rate

Absorbed flux is $F_a = 0.19 \times 342\ \mathrm{Wm^{-2}} = 65\ \mathrm{Wm^{-2}}$

Mass of atmosphere $M = \frac{P_0}{g} = \frac{1.013 \times 10^5\ \mathrm{kg\ m\ sec^{-2}\ m^{-2}}}{9.8\ \mathrm{m\ sec^{-2}}}$

$= 10^4\ \mathrm{kg/m^2}$

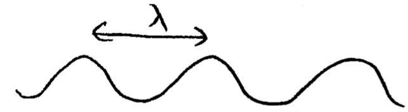
To convert absorbed flux to a heating rate $\frac{dT}{dt}$:

$$F_a = M c_p \frac{dT}{dt}$$

$$\frac{dT}{dt} = \frac{F_a}{M c_p} = \frac{65\ \mathrm{Wm^{-2}}}{10^4\ \mathrm{kg\ m^{-2}} \times 1004\ \mathrm{J\ kg^{-1}\ K^{-1}}}$$

$$= 6.3 \times 10^{-6}\ \frac{\mathrm{K}}{\mathrm{sec}} = 0.54\ \frac{\mathrm{K}}{\mathrm{day}}$$

Petty 2.2   3 points.   Doppler shift.



waves are emitted at frequency ν,

time interval $\Delta t = \frac{1}{\nu}$, wavelength $\lambda = \frac{c}{\nu}$ if source and detector are stationary.

If source is moving away at velocity v, then in time interval Δt the source travels a distance $v\Delta t = \frac{v}{\nu}$.

So the wave-crest is retarded: spacing between wave crests is $\lambda' = \lambda + \frac{v}{\nu}$.

The time for the wave-crest to travel the extra distance $(\lambda' - \lambda$ $\frac{v}{\nu}$ (at speed c) is $\frac{v}{\nu c}$.

So the time between successive wave-crests is $\frac{1}{\nu} + \frac{v}{\nu c}$.

Frequency is the reciprocal of this:

$$\nu' = \left(\frac{1}{\nu} + \frac{v}{\nu c}\right)^{-1} = \frac{\nu c\,(c-v)}{c^2 - v^2} \underset{v \ll c}{\approx} \frac{\nu}{c}(c-v) = \nu\left(1 - \frac{v}{c}\right)$$

$$\Delta\nu = \nu' - \nu = -\frac{\nu v}{c}$$

[We have ignored relativistic effects (time-dilation).]

Petty 2.3 5 points Boundary layer heating rate

(a) $F(t) = F_o \cos\left(\frac{\pi(t-12)}{12}\right)$

$$\int F(t)\,dt = F_o \int_0^{18} \cos \frac{\pi(t-12)}{12}\,dt$$, where the integral has units of hours.

$$x \equiv \frac{\pi(t-12)}{12} \; ; \; dx = \frac{\pi\,dt}{12}$$

t	x
6	$-\pi/2$
18	$\pi/2$

$$= F_o \frac{12}{\pi} \int_{-\pi/2}^{\pi/2} \cos x\,dx$$

$$= 500\,\mathrm{Wm^{-2}} \underbrace{\left(\frac{12}{\pi}\cdot 2\right)\mathrm{hr}}_{7.6\,\mathrm{hr.}} = 3820 \frac{\mathrm{watt\text{-}hours}}{\mathrm{m^2}}$$

To convert to Joules:

$$3820 \frac{\mathrm{J}}{\mathrm{sec}} \cdot \frac{\mathrm{hr}}{\mathrm{m^2}} \times \frac{3600\,\mathrm{sec}}{\mathrm{hr}} = 1.38 \times 10^7 \frac{\mathrm{J}}{\mathrm{m^2}}$$ daily total

(b) Atmospheric mass in boundary layer: $M = \rho_a \Delta z$

Temperature change ΔT:

$$\rho_a \Delta z\, C_p \Delta T = 1.38 \times 10^7\,\mathrm{J\,m^{-2}}$$

$$\Delta T = \frac{1.38 \times 10^7\,\mathrm{J\,m^{-2}}}{1\,\mathrm{kg\,m^{-3}}\;\; 1004\,\mathrm{J\,kg^{-1}\,K^{-1}} \times 1000\,\mathrm{m}} = 13.7\,\mathrm{K/day}$$

(c) If boundary layer is only 10 m deep, heating rate is a factor of 100 larger: $\Delta T = 1370\,\mathrm{K/day}$.

Convection will cause boundary layer to deepen.

Petty 2.9 3 points. Light source on the Moon

(a) Let n = number of photons emitted per second.

$$1\,W = \frac{1\,J}{sec} = nh\nu = \frac{nhc}{\lambda}$$

$$n = \frac{1\ J\ sec^{-1} \times 0.5\times10^{-6}\,m}{6.63\times10^{-34}\ J\cdot sec \times 3\times10^{8}\ m\ sec^{-1}} \approx 2.5\times10^{18}\ \frac{photons}{sec}$$

(b) These photons spread out in all directions.
At a distance of D_m they are spread over an area $4\pi D_M^2$.
The telescope of radius r presents an area πr^2, so it intercepts a fraction $\frac{\pi r^2}{4\pi D_M^2}$ of the emitted photons.

$$\frac{(2.5\times10^{18}\ photons\ sec^{-1})\ \pi\cdot 100\ cm^2}{4\pi\ (3.84\times10^{5}\ km)^2\times 10^{10}\ cm^2/km^2} = 0.04\ photons/sec.$$

$$= 2.6\ photons/minute,$$

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① Petty 2.12 Eclipses (4 points)

(a) Angular radii

		angular radius (θ_{max})	angular diameter
Moon	$\frac{R}{D} = \frac{1.74\times10^3}{3.84\times10^5} = 0.00453$ radians $=$	$0.25962°$	$0.51924°$
Sun	$\frac{R}{D} = \frac{6.96\times10^5}{1.496\times10^8} = 0.00465$ radians $=$	$0.26656°$	$0.53312°$

(b) Solid angle

$$\Delta\omega = \int_0^{2\pi}\int_0^{\theta_{max}} \sin\theta \, d\theta \, d\phi$$

$$= 2\pi\left[\cos 0 - \cos\theta_{max}\right]$$

$$= 2\pi\left[1 - \cos\theta_{max}\right] = \begin{cases} 6.45\times10^{-5} \text{ sr} & \text{(Moon)} \\ 6.80\times10^{-5} \text{ sr} & \text{(Sun)} \end{cases}$$

(c) The Sun is larger, by 5%

(d) Total solar eclipses would not be possible.
Total solar eclipses occur when the Moon is closer to the Earth than its average distance.

Eccentricity of lunar orbit is 7%

② Solar flux at perihelion & aphelion (3 points)

L = solar luminosity

r = Earth-sun distance

$\bar{r}$ = average Earth-sun distance

e = eccentricity

$r_a = \bar{r}(1+e)$
$r_p = \bar{r}(1-e)$
subscripts p = perihelion, a = aphelion.

$$S_p = \frac{L}{4\pi r_p^2} \qquad S_a = \frac{L}{4\pi r_a^2}$$

so $$\frac{S_p}{S_a} = \frac{r_a^2}{r_p^2} = \frac{\bar{r}^2(1+e)^2}{\bar{r}^2(1-e)^2} = \left(\frac{1+e}{1-e}\right)^2$$

Earth $e = 0.0175$

$\frac{S_p}{S_a} = 1.07$ i.e. 7% larger in January than in July

Mars $e = 0.093$

$\frac{S_p}{S_a} = 1.45$ i.e. 45% larger at perihelion than at aphelion.

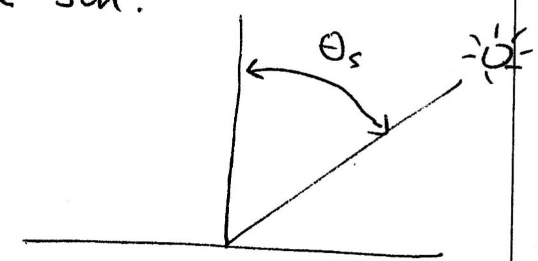
③ Relative brightness of snow & sun (4 points)

ⓐ $f = \frac{\Delta\omega}{2\pi} = \frac{6.80 \times 10^{-5}\ sr}{2\pi\ sr} = 1.08 \times 10^{-5}$

ⓑ Intensity from Sun is I_{sun} (average intensity across the solar disk)
Flux received from Sun by a horizontal snowfield on Earth's surface:

$$F\downarrow = \int I_{sun} \cos\theta \, d\omega \approx I_{sun} \cos\theta_s \int d\omega$$

because θ doesn't vary much across the Sun.



So $F\downarrow = I_{sun} \cos\theta_s \, \Delta\omega$.
This flux is incident on the snow.

Reflected flux:

$F\uparrow(\text{snow}) = A \cdot F\downarrow = A \, I_{sun} \, \Delta\omega \cos\theta_s$, where A = albedo.

Assume snow is an isotropic reflector.

Then $I_{snow} = \frac{F\uparrow(\text{snow})}{\pi} = \frac{A \, I_{sun} \, \Delta\omega \cos\theta_s}{\pi}$.

and $\frac{I_{snow}}{I_{sun}} = \frac{A \, \Delta\omega \cos\theta_s}{\pi} \underset{\theta_s = 60^\circ}{=} \frac{A \, \Delta\omega}{2\pi} = f \cdot A$

If $A = 1$, the brightness ratio of snow to Sun is the same as the fraction of the sky occupied by the Sun.
(if the Sun is at an "average" zenith angle of 60°).

ⓒ $\frac{I_{snow}}{I_{sun}} = f = 5^{-n}$

$n = 7.1$ sunglasses needed in addition to the one for snow.

$n + 1 = 8.1$ (or 9 to be safe!)

④ Relative densities of molecules and photons (5 points)

(Here we are considering only solar photons, not longwave photons emitted by earth.)

Let $S_o \equiv$ solar constant ; $A \equiv$ albedo.

ⓐ Downward flux for zenith sun $= S_o = I_{sun}\,\Delta\omega_{sun}$

Assume thickness of atmosphere is small compared to radius of Earth, so the upward photons all come from nearby, where the sun is still nearly overhead.

Upward flux $= A S_o$

Upward intensity $= A S_o / \pi$ (isotropic in upward hemisphere)

Mean intensity $\bar{I} = \frac{1}{4\pi}\left[\frac{A S_o}{\pi} \times 2\pi + \underbrace{I_{sun}\,\Delta\omega_{sun}}_{S_o}\right]$

$$\bar{I} = \frac{1}{4\pi}\left[2 A S_o + S_o\right] = \frac{S_o}{4\pi}(1+2A)$$

Energy density $u = \frac{4\pi}{c}\bar{I}$ and $u = n_p h\nu = \frac{n_p h c}{\lambda}$

where $n_p =$ number-density of photons.

$$\therefore\ n_p = \frac{4\pi \bar{I} \lambda}{c^2 h} = \frac{S_o (1+2A)\lambda}{c^2 h}$$

$$= \frac{(1370\ \mathrm{Wm^{-2}})(1+0.6)(0.5\times10^{-6}\,\mathrm{m})}{(3\times10^{8}\ \mathrm{m/sec})^2 \times 6.626\times10^{-34}\ \mathrm{J \cdot sec}} = \underline{1.84\times10^{13}\ \frac{\text{photons}}{\mathrm{m^3}}}$$

ⓑ Number-density of molecules n_m

Ideal gas law $p = n_m k T$, where

k is Boltzmann's constant, $1.38\times10^{-23}\ \mathrm{J\,K^{-1}\,molecule^{-1}}$

For temperature $T = 273\,\mathrm{K}$ and pressure $p = 101325\ \mathrm{Pa}$,

$n_m = 2.7\times10^{25}$ molecules/$\mathrm{m^3}$ at sea level.

So at sea level $\frac{n_m}{n_p} = \frac{2.7\times10^{25}}{1.8\times10^{13}} = e^{28}$.

[In the lower atmosphere photons are far outnumbered by molecules.]

So $n_m = n_p$ at 28 scale heights, or $\underline{220\ \mathrm{Km}}$

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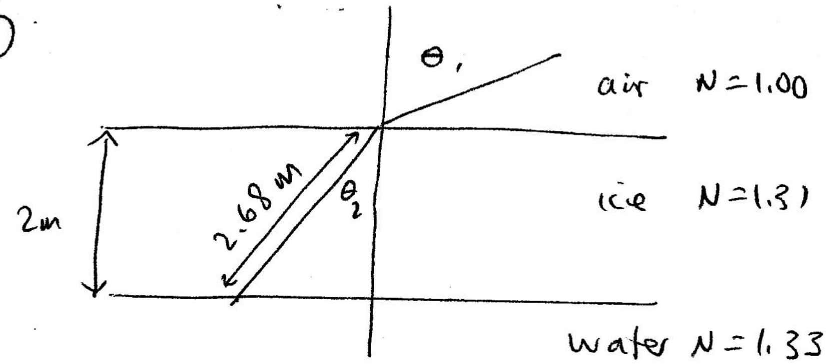
① Transmission of sunlight through ice (3 points)

(a)



$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}$$ $\theta_1 = 60°$; $\theta_2 = 41.4°$, path length $2\sec\theta_2 = 2.67\,m$.

(b) There is negligible reflection at the ice - water interface

Reflection at air - ice interface:

$$r_\perp = \left[\frac{\cos\theta_1 - n_2\cos\theta_2}{\cos\theta_1 + n_2\cos\theta_2}\right]^2 = 0.104$$

$$r_\parallel = \left[\frac{\cos\theta_2 - n_2\cos\theta_1}{\cos\theta_2 + n_2\cos\theta_1}\right]^2 = 0.005$$

Reflectance for (unpolarized) sunlight $r = \frac{r_\perp + r_\parallel}{2} = 0.054$

$$\text{Transmittance} = (1-r)\,e^{-2\beta_a\sec\theta_2} = \begin{cases} 0.943 & \text{blue} \\ 0.237 & \text{red} \end{cases}$$

Ratio $\frac{\text{red}}{\text{blue}} = 0.25$, so transmitted light looks <u>blue</u>

② Units of Planck function.

At $\lambda = 500\,nm$ $\quad \pi B_\lambda = 2000\ W m^{-2} \mu m^{-1}$

$$\frac{d\nu}{\nu} = -\frac{d\lambda}{\lambda} \quad \text{so} \quad \lambda B_\lambda = \nu B_\nu$$

$$\pi \lambda B_\lambda = (2000\ W m^{-2} \mu m^{-1})(0.5\ \mu m) = 1000\ W m^{-2}.$$

$\lambda = 500\,nm$ corresponds to $\nu = 6 \times 10^{14}\ Hz$; $\bar{\nu} = 20{,}000\ cm^{-1}$

$$(1000\ W m^{-2}) / (6 \times 10^{14}\ Hz) = 1.67 \times 10^{-12}\ W m^{-2} Hz^{-1}$$

$$(1000\ W m^{-2}) / (20{,}000\ cm^{-1}) = 0.05\ W m^{-2} (cm^{-1})^{-1}$$

③ Maxima of the Planck function

(a) $$B_\lambda = \frac{c_1}{\lambda^5 \left[e^{c_2/\lambda T} - 1\right]}$$

$$\frac{dB_\lambda}{d\lambda} = c_1(-5)\lambda^{-6}\left[e^{c_2/\lambda T} - 1\right]^{-1} + c_1 \lambda^{-5}(-1)\left[e^{c_2/\lambda T} - 1\right]^{-2} e^{c_2/\lambda T}\left(\frac{c_2}{T}\right)(-\lambda^{-2})$$

At max, $$\frac{dB_\lambda}{d\lambda} = -5\lambda + \frac{c_2}{T}\left(\frac{e^{c_2/\lambda T}}{e^{c_2/\lambda T} - 1}\right) = 0$$

or $$e^{-c_2/\lambda T} = 1 - \frac{c_2}{5\lambda T}, \quad \text{or} \quad \boxed{e^{-a} = 1 - \frac{a}{5}} \quad \text{where } a = \frac{c_2}{\lambda T}.$$

This can be solved graphically to give $a = 4.9651$.

$$c_2 = \frac{hc}{k} = 1.439\ cm\,K, \quad \text{so} \quad \lambda T = \frac{c_2}{a} = \underline{0.290\ cm\,K}.$$

(b) $$B_{\bar{\nu}} = \frac{c_1 \bar{\nu}^3}{e^{c_2 \bar{\nu}/T} - 1}$$

$$0 = \frac{dB_{\bar{\nu}}}{d\bar{\nu}} = c_1(3\bar{\nu}^2)\left[e^{c_2\bar{\nu}/T} - 1\right]^{-1} + c_1\bar{\nu}^3(-1)\left[e^{c_2\bar{\nu}/T} - 1\right]^{-2}\left[e^{c_2\bar{\nu}/T}\right]\left(\frac{c_2}{T}\right)$$

so $$3 - \frac{\bar{\nu} c_2}{T} e^{c_2\bar{\nu}/T}\left[e^{c_2\bar{\nu}/T} - 1\right]^{-1} = 0\ ;$$

$$e^{-c_2\bar{\nu}/T} = 1 - \frac{\bar{\nu}c_2}{3T}, \quad \text{or} \quad \boxed{e^{-a} = 1 - \frac{a}{3}} \quad \text{where } a = \frac{\bar{\nu}c_2}{T}.$$

This can be solved graphically to give $a = 2.8214$.

But $c_2 = 1.439$ cm K, so $\frac{T}{\bar{\nu}} = \frac{c_2}{a} = \underline{0.510 \text{ cm K}}$

Thus the planck function peaks at a different value of λT depending on whether B_ν or B_λ is plotted,

because $B_\nu = \frac{dB}{d\nu} \neq \frac{dB}{d\lambda} = B_\lambda$

However, if the Planck function is plotted instead as $\frac{dB}{d\log\lambda}$, it is the same as $\frac{-dB}{d\log\nu}$,

because $d\log\lambda = -d\log\nu$.

Function	Transcendental equation for peak	Value of λT at peak (cm·K)	Peak value of λ (μm) at T = 6000K	at T = 255K	at T = 288K
$\frac{dB}{d\lambda}$	$e^{-a} = 1 - \frac{a}{5}$	0.290	0.48	11.4	10.1
$\frac{dB}{d\nu}$	$e^{-a} = 1 - a/3$	0.510	0.85	20.0	17.7
$\frac{dB}{d\ln\lambda}$	$e^{-a} = 1 - a/4$	0.367	0.61	14.4	12.7
$\frac{dB}{d\ln\nu}$	$e^{-a} = 1 - a/4$	0.367	0.61	14.4	12.7

④ Petty 6.14.

$$B_\lambda = \frac{2hc^2}{\lambda^5 \left[e^{hc/\lambda kT} - 1\right]} \qquad \frac{2hc^2}{\lambda^5} = 479 \ \mathrm{Wm^{-2}\,\mu m^{-1}\,sr^{-1}}$$

$$\lambda = 12\ \mu m = 12\times10^{-6}\ m \qquad \frac{hc}{\lambda k} = 1200\ K.$$

$$\frac{hc}{\lambda kT} = 0.200 \text{ for Sun, } 4.00 \text{ for Earth.}$$

	$e^{hc/\lambda kT} - 1$	emitted intensity $Wm^{-2}sr^{-1}\mu m^{-1}$	incident solar flux $Wm^{-2}\mu m^{-1}$	flux reflected or emitted $Wm^{-2}\mu m^{-1}$
Sun:	0.221	2167	0.147	0.0147
Earth:	53.5	8.95×0.9		25.3 ($=\pi\times8.95\times0.9$)

ratio $\dfrac{\text{reflected solar radiation}}{\text{emitted terrestrial radiation}} = \dfrac{0.0147}{25.3} = 5.8\times10^{-4} = \underline{\underline{0.06\%}}$

ATMS 532 Homework 4 answers.

(1)

(a) $$B_\nu = \frac{2h\nu^3}{c^2\left(e^{h\nu/kT}-1\right)}$$ (6 points)

$$\frac{h\nu}{kT} \ll 1 \quad \text{so} \quad e^{h\nu/kT}-1 \approx 1+\frac{h\nu}{kT}-1 = \frac{h\nu}{kT}$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2 \cdot \frac{h\nu}{kT}} = \frac{2\nu^2 kT}{c^2},$$ i.e. B is proportional to T if $\frac{h\nu}{kT} \ll 1$.

$$B_\nu(T_B) = \frac{2\nu^2 kT_B}{c^2} = \epsilon B(T) = \frac{\epsilon\, 2\nu^2 kT}{c^2} \quad \text{so} \quad \underline{\underline{T_B = \epsilon T}}$$

(b) $\epsilon B_\lambda(T) = B_\lambda(T_B)$ [can do this using either B_λ or B_ν]

$$\frac{\epsilon\, 2hc^2\lambda^{-5}}{e^{hc/\lambda kT}-1} = \frac{2hc^2\lambda^{-5}}{e^{hc/\lambda kT_B}-1}$$

Solving for T_B: $$\frac{1}{T_B} = \frac{\lambda k}{hc}\ln\left[\frac{e^{hc/\lambda kT}-1+\epsilon}{\epsilon}\right]$$

Plugging in $T = 273\,K$, $\lambda = 1.55$ cm, $\epsilon = 0.43$,

and $\frac{hc}{k} = 1.4413$ cm K, we find $\underline{\underline{T_B = 117.65\,K}}$.

This is the true value of T_B we will obtain from the radiance measurement. Now if we use it to infer T simply from $T_B \approx \epsilon T$,

we obtain $T = 273.6$, an error of $\underline{\underline{+0.6\,K}}$

(3 points)

(2) The reflectivity of the ground for infrared radiation is $r_g = 1 - \epsilon_g$, by Kirchhoff's law. The upward radiation is the sum of the reflected and emitted radiation.

emitted : $\epsilon_g \sigma T_g^4$ reflected : $r_g \sigma T_c^4$

so $\epsilon_g \sigma T_g^4 + (1-\epsilon_g) \sigma T_c^4 = 1.05 \sigma T_g^4$.

solving for T_c : $\frac{T_c}{T_g} = \left(\frac{1.05 - \epsilon_g}{1 - \epsilon_g}\right)^{1/4} = 1.11$

Therefore $\underline{\underline{T_c = 277\ K}}$.

(3 points)

(3) $F_B = \sigma T^4$. $\frac{dF_B}{dT} = 4\sigma T^3$

$15°C = 288K$, $\frac{dF_B}{dT} = 5.4\ Wm^{-2}K^{-1}$

$-18°C = 255K$, $\frac{dF_B}{dT} = 3.76\ Wm^{-2}K^{-1}$.

outgoing longwave radiation (OLR):

$\frac{d\ OLR}{dT_s} \approx 1.8\ Wm^{-2}K^{-1}$. It's smaller because of water-vapor feedback. In warmer atmosphere there is more water vapor, atmosphere becomes more opaque in longwave, emission to space is from a higher ~~level~~, colder level, so temperature of emitting level does not increase as much as surface temperature.

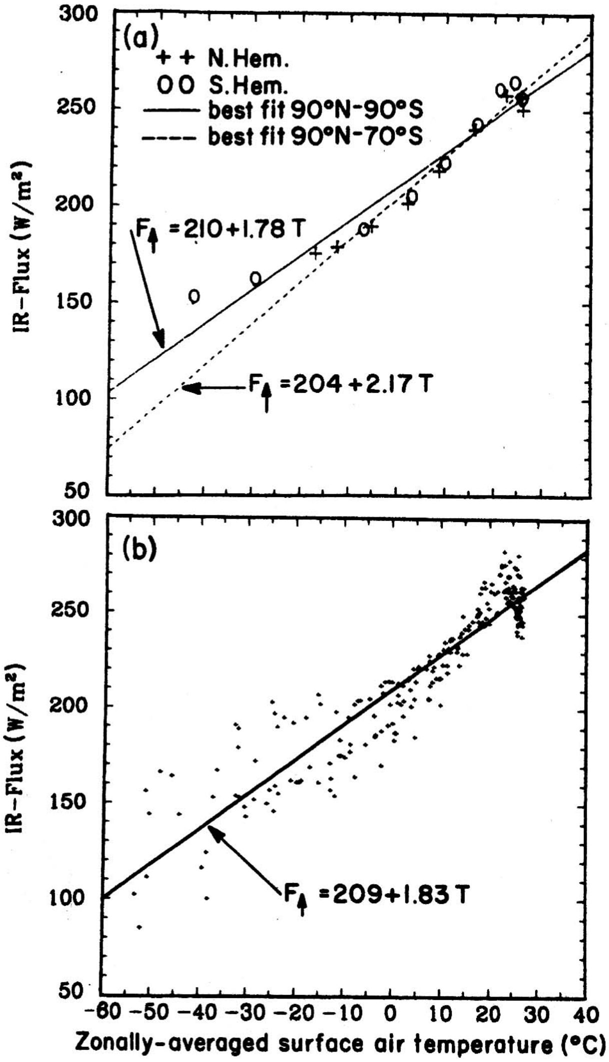


FIG. 5. Outgoing infrared flux as a function of surface temperature for 10° wide latitude zones. IR flux data are from Ellis and Vonder Haar (1976).
(a) Mean annual values. Plus marks indicate Northern Hemisphere; circles, Southern Hemisphere. Solid line: best fit to all points; dashed line: best fit to all points excluding 70–90°S. (b) Monthly values. Points for all 12 months for each of 18 latitude zones are plotted. The line is the least-squares fit to these points.

Warren & Schneider
J. Atmos. Sci. 1979
vol 36 p. 1377

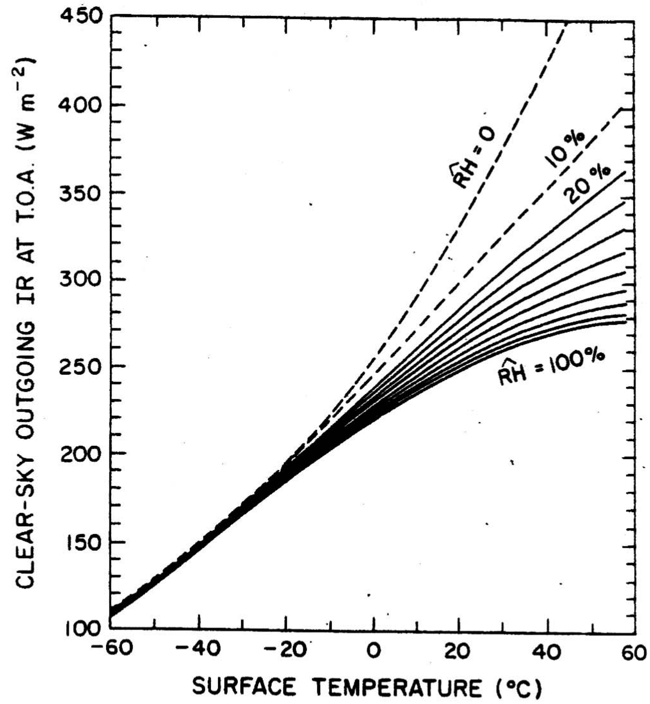


FIG. 9. Family of curves depicting the parameterization of clear-sky outgoing IR irradiance as a function of T_s and $\widehat{RH}$. Dashed lines for $\widehat{RH} = 0$ and 10% are not used in deriving the parameterization function but are shown for the sake of completeness.

Thompson & Warren 1982
J. Atmos. Sci. 39, 2667

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3 points each, total 15 points

1 Petty 7.6 page 173. cloud.

a liquid in cloud

$\rho_w = 0.1 \text{ g m}^{-3}$ $\quad k_s = 150 \frac{m^2}{kg} = 0.15 \text{ m}^2 \text{g}^{-1}$

$\beta_s = k_s \rho_w = \left(0.15 \frac{m^2}{g}\right)\left(0.1 \frac{g}{m^3}\right) = \underline{0.015 \text{ m}^{-1}}$

vapor in cloud

$\beta_a = 10 \text{ km}^{-1} = \underline{0.010 \text{ m}^{-1}}$

cloud (mixture of liquid & vapor):

$\beta_e = \beta_a + \beta_s = \underline{\underline{0.025 \text{ m}^{-1}}}$

$\tilde{\omega} = \frac{0.015}{0.025} = \underline{\underline{0.6}}$

b $\tau = \beta_e \Delta z = (0.025 \text{ m}^{-1})(100 \text{ m}) = 2.5$

c $\theta = 60^\circ$

$\tau \sec\theta = 5.0$

$\frac{I_{bot}}{I_{top}} = e^{-5} = 0.007$

2 Petty 7.7 page 179. UV absorption by O_2

(a) atmospheric mass per unit area:

$$M = \frac{p_0}{g} = 1.01\times10^{5}\ \frac{kg\ m}{sec^2\ m^2} \times \frac{sec^2}{9.8\ m} = \boxed{1.03\times10^{4}\ \frac{kg\ air}{m^2}}$$

(b) oxygen. Molecules of O_2 per unit area:

$$\int N(z)\,dz = 1.03\times10^{4}\ \frac{kg\ air}{m^2} \times \frac{kmol\ air}{29\ kg\ air} \times \frac{0.21\ kmoles\ O_2}{Kmole\ air}$$

$$\times\ 6.023\times10^{26}\ \frac{molecules\ O_2}{kmole\ O_2}$$

$$\int N(z)\,dz = \boxed{4.5\times10^{28}\ \frac{molecules\ O_2}{m^2}}$$

(c) $$\tau = \sigma\int N(z)\,dz = \left(7\times10^{-29}\ \frac{m^2}{molecule}\right)\times\left(4.5\times10^{28}\ \frac{molecules}{m^2}\right)$$

$$= 3.15$$

vertical transmittance $e^{-\tau} = 0.043$

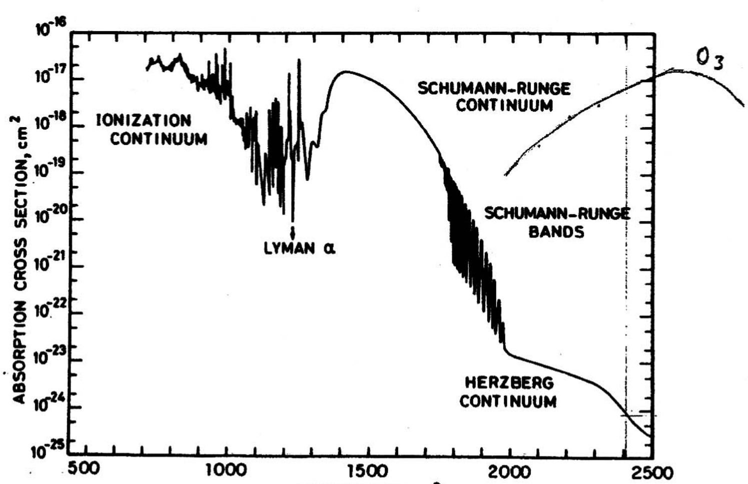


FIG. 5.4. Absorption cross section of $^{16}O^{16}O$ in the ultraviolet spectrum. After Brasseur and Solomon (1984).

Goody & Yung 1989

③ UV absorption by ozone.

ⓐ Gas density at STP : use ideal gas law. $\rho = \frac{P}{RT}$

$$\rho = \frac{101325 \ \mathrm{N\,m^{-2}}}{(8.37 \ \mathrm{J\,mole^{-1}\,K^{-1}})(273\mathrm{K})} = 44.6 \ \mathrm{moles/m^3}.$$

Ozone column amount :

$$0.003 \ \mathrm{m} \times 44.6 \ \frac{\mathrm{moles}}{\mathrm{m^3}} = 0.134 \ \frac{\mathrm{moles}}{\mathrm{m^3}} = \boxed{8.07 \times 10^{22} \ \frac{\mathrm{molecules}}{\mathrm{m^3}}}$$

ⓑ $$\tau^* = \left(8.07\times10^{22} \ \frac{\mathrm{molecules}}{\mathrm{m^2}}\right)\left(8.2\times10^{-22} \ \frac{\mathrm{m^2}}{\mathrm{molecule}}\right) = \underline{\underline{66}}$$

ⓒ vertical transmittance is $e^{-\tau^*} = 2\times10^{-29}$

④ Petty 7.11 $\tau^* = \frac{3}{4}\frac{Q_e L}{\rho_l r}$ derived in class

where L is liquid water path
ρ_l is density of pure liquid water (1 g / cm³)
r is droplet radius
$Q_e \approx 2$ for $r \gg \lambda$.

Mass of droplet $m = \rho_l \frac{4}{3}\pi r^3$

$$L = H \cdot N \cdot m = H N \rho_l \frac{4}{3}\pi r^3, \quad \text{so} \quad \frac{1}{r^3} = \frac{NH}{L}\rho_l \frac{4}{3}\pi$$

$$(\tau^*)^3 = \left(\frac{3}{4}\frac{Q_e L}{\rho_l}\right)^3 \cdot \frac{NH}{L}\rho_l \frac{4}{3}\pi$$

$$\text{so} \quad \tau^* = Q_e \left[\frac{9 L^2 N H \pi}{16 \rho_l^2}\right]^{1/3}$$

5 Petty 7.12 page 200.

$$\tau^* = Q_e \left[\frac{9\pi L^2 H}{16 \rho_l^2} N \right]^{1/3}$$

$Q_e = 2$ $L = 0.01 \ kg m^{-2}$ $H = 100 \ m$

$$\frac{9\pi L^2 H}{16 \rho_l^2} = \frac{9\pi \, (0.01 \ kg m^{-2})^2 \times 100 m}{16 \, (1000 \ kg m^{-3})^2}$$

$$= 1.767 \times 10^{-8} \ m^3 = 0.0177 \ cm^3$$

$N \ (cm^{-3})$	τ^*	τ^*/μ	$t_{dir} = e^{-\tau^*/\mu}$
100	2.42	4.84	7.9×10^{-3}
1000	5.21	10.42	3.0×10^{-5}

---

we can also compute effective radius of cloud drops:

$LWP = 0.01 \ \frac{kg}{m^2} = 10 \ g/m^2$

$H = 100 \ m$

$LWC = 0.1 \ \frac{g}{m^3} = N \cdot \frac{4}{3} \pi r^3 \rho_l$

$\rho_l = 10^6 \ g \ m^{-3}$

so $r = 2.9 \times 10^{-6} m = 2.9 \ \mu m$ for $N = 1000 \ cm^{-3}$

$6.2 \ \mu m$ for $N = 100 \ cm^{-3}$

ATMS 532 Problem Set 6 answers.

① Let N_0 be the total number of photons starting at optical depth zero ($\tau = 0$). The fraction of photons reaching distance (optical depth) τ_i is

$$N/N_0 = e^{-\tau_i}.$$

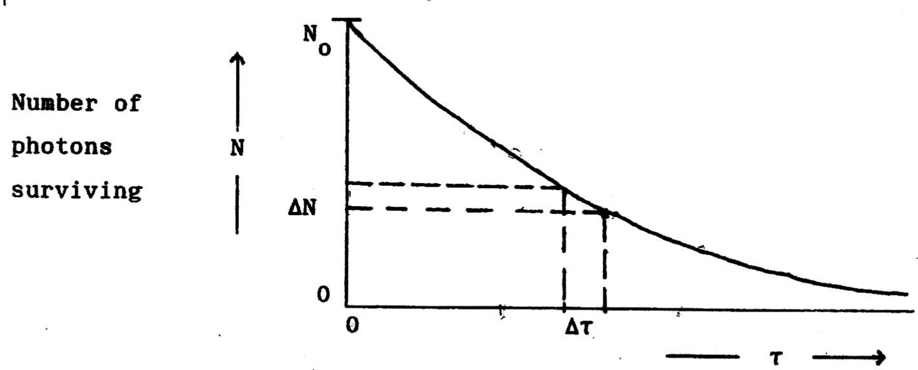
The fraction reaching distance $\tau_i + \Delta\tau$ is $e^{-(\tau_i + \Delta\tau)}$.

The fraction $\Delta N_i / N_0$ extinguished between τ_i and $\tau_i + \Delta\tau$ is therefore

$$\frac{\Delta N_i}{N_0} = e^{-\tau_i} - e^{-(\tau_i + \Delta\tau)} = e^{-\tau_i} - e^{-\tau_i} \underbrace{e^{-\Delta\tau}}_{\approx 1 - \Delta\tau}$$

$$= e^{-\tau_i} - e^{-\tau_i} + \Delta\tau e^{-\tau_i}$$

$$= \Delta\tau e^{-\tau_i}$$



The mean free path is

$$\frac{\Sigma\, \Delta N_i\, \tau_i}{\Sigma\, \Delta N_i} = \frac{\Sigma\, \Delta\tau e^{-\tau_i}\, \tau_i}{\Sigma\, \Delta\tau e^{-\tau_i}} = \frac{\int \tau e^{-\tau}\, d\tau}{\int e^{-\tau}\, d\tau} = \frac{1}{1} = 1$$

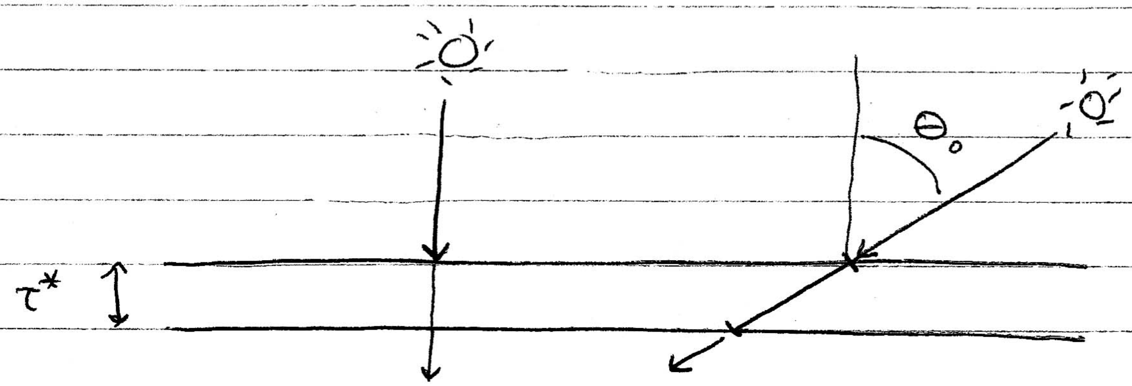
Numerator: $$\left[\int_0^\infty \tau e^{-\tau}\, d\tau = \tau e^{-\tau} + e^{-\tau} \right]_\infty^0 = 0 + 1 - 0 - 0$$

To show that $\lim_{\tau \to \infty} (\tau e^{-\tau}) = 0$, show that $\frac{\tau}{e^\tau} \to 0$ or $\frac{e^\tau}{\tau} \to \infty$:

$$\frac{e^\tau}{\tau} = \frac{1}{\tau}\left(1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \ldots\right) = \frac{1}{\tau} + 1 + \frac{\tau}{2!} + \frac{\tau^2}{3!} + \ldots \rightarrow \infty.$$

So the mean free path is one unit of optical depth.

② Absorption of solar energy by a thin layer.



	overhead sun	oblique sun
	$\theta_o = 0°$	$\theta_o > 0°$
incident flux ⊥ beam	S_o	S_o
flux on horizontal surface at T.O.A.	S_o	$S_o \cos\theta_o = S_o \mu$
transmittance of absorbing layer	$e^{-\tau^*} \approx 1-\tau^*$	$e^{-\tau^*/\mu} \approx 1 - \frac{\tau^*}{\mu}$
absorptance	τ^*	τ^*/μ
absorbed flux	$S_o \tau^*$	$S_o \mu \frac{\tau^*}{\mu} = S_o \tau^*$

∴ Absorbed flux is independent of solar zenith angle, provided that $\frac{\tau^*}{\mu} \ll 1$.

(3) Survival probability of a photon after N extinction events is $\tilde{\omega}^N$.

<u>Questions</u> we might ask of the plot: $P = \tilde{\omega}^N$

Given $\tilde{\omega}$, P, what is N? e.g. for P = 0.8, 0.5, 0.2 ...

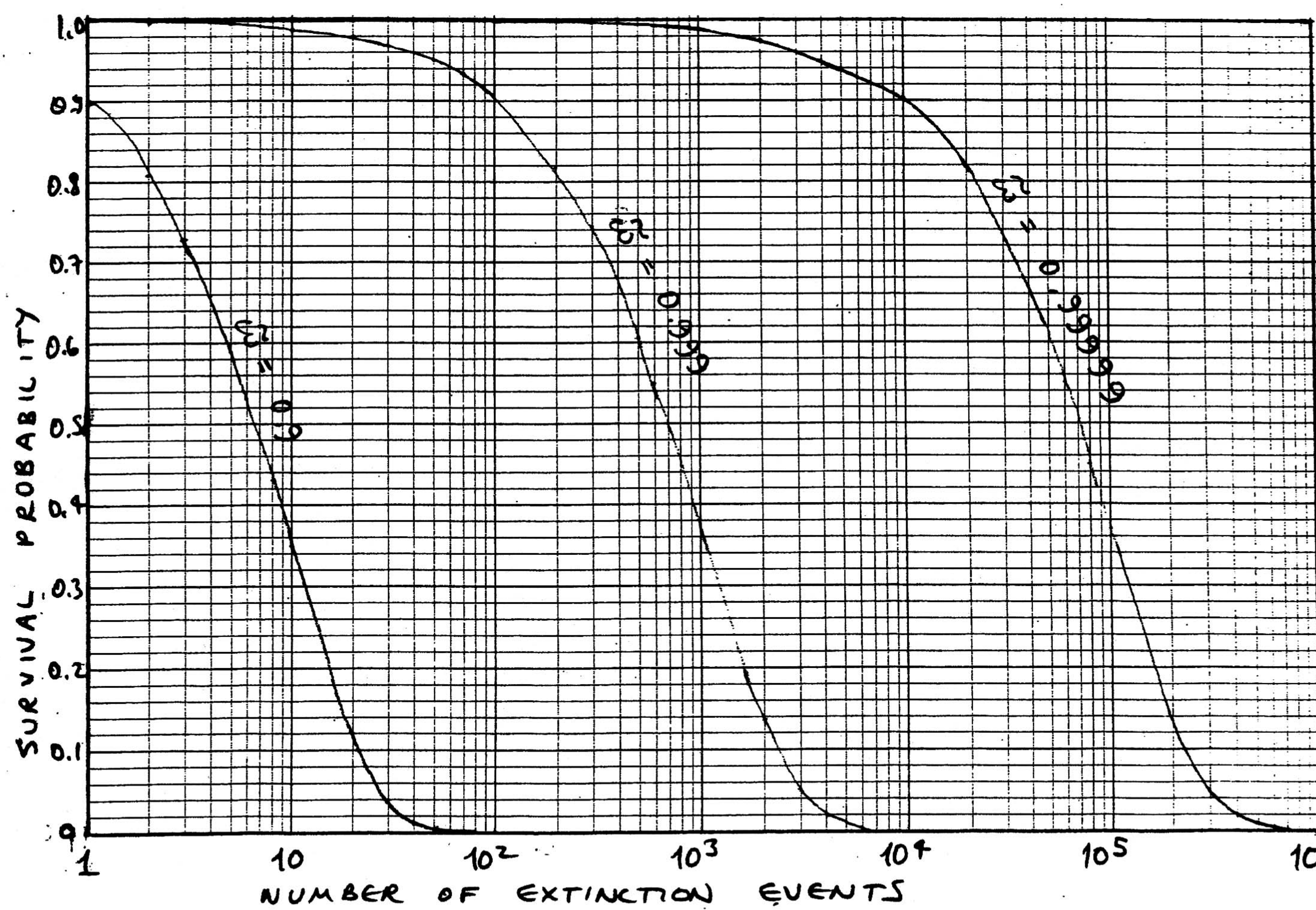
Given $\tilde{\omega}$, N, what is P? eg. for N = 10, 100, 1000, 10000.

<u>Criteria for plotting</u>

(a) All curves should fit on one frame

(b) values of P go from 1.0 to 0.0

The intervals (0, 0.1) and (0.9, 1.0) are equally important if our concern is energy fluxes.



① Show that $\frac{d E_n(\tau)}{d\tau} = -E_{n-1}(\tau)$.

2 points

$$E_n(\tau) = \int_1^\infty \frac{e^{-x\tau}}{x^n}\,dx$$

$$\frac{d E_n(\tau)}{d\tau} = \int_1^\infty \frac{-x e^{-x\tau}}{x^n}\,dx = -\int_1^\infty \frac{e^{-x\tau}}{x^{n-1}}\,dx = -E_{n-1}(\tau)$$

② $2E_3(\tau) = e^{-\beta\tau}$.

4 points

To show: $\beta \to 2$ as $\tau \to 0$; $\beta \to 1$ as $\tau \to \infty$.

(a) $\tau \to 0$.

Taylor expansion: $E_3(\tau) \simeq E_3(0) + (\tau - 0)\frac{d}{d\tau}E_3(\tau)\Big|_{\tau=0}$

$$= \frac{1}{2} + \tau\left[-E_2(0)\right] = \frac{1}{2} - \tau$$

So $2E_3(\tau) = 1 - 2\tau$. Also, $e^{-\beta\tau} \approx 1 - \beta\tau$ for small τ,

so $\beta = 2$ or $\theta = 60°$

(b) $\tau \to \infty$.

$$2E_3(\tau) = e^{-\beta\tau}$$

$$\ln 2 + \ln E_3(\tau) = -\beta\tau$$

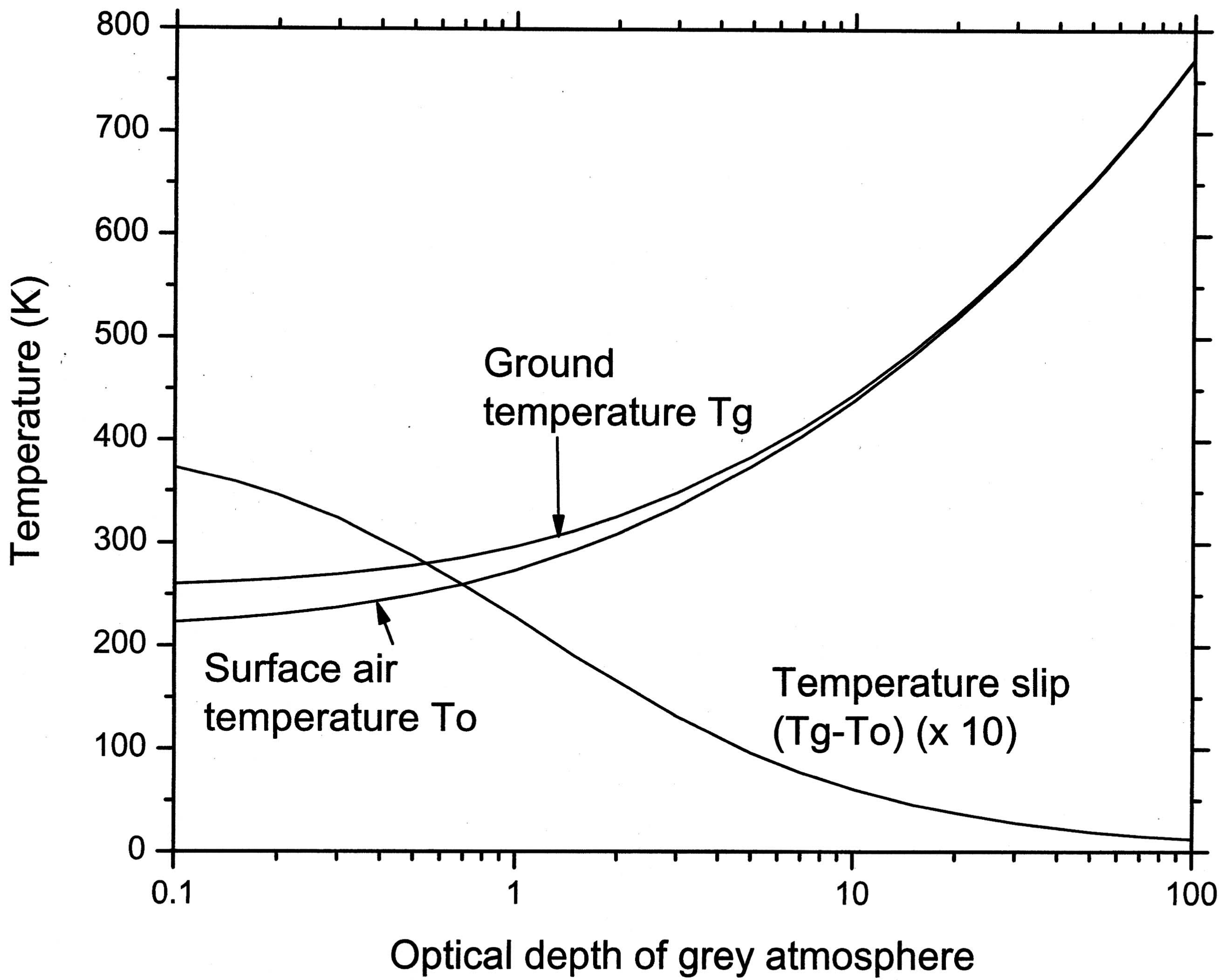
so $\beta = \left[-\ln 2 - \ln E_3(\tau)\right]/\tau$.

Use asymptotic form, (Abramowitz & Stegun, p. 231):

$E_n(\tau) \to e^{-\tau}/\tau$ as $\tau \to \infty$ for all n.

so $$\beta \approx \frac{-\ln 2 + \tau + \ln\tau}{\tau}.$$

As $\tau \to \infty$, $\tau \gg \ln\tau$, so $\beta \to \frac{\tau}{\tau} = 1$. or $\theta = 0°$


800
700
600
500
400
300
200
100
0
Temperature (K)
0.1
1
10
100
Optical depth of grey atmosphere
Ground
temperature Tg
Surface air
temperature To
Temperature slip
(Tg-To) (x 10)

3 points

3

(4) Excited state populations.

$$\frac{N_1}{N_0} = \frac{g_1}{g_0} e^{-\Delta E/kT} \quad \text{where } \Delta E = \left(\frac{\nu}{c}\right) hc$$

2 points

mode	g_1/g_0	$\Delta E/k$ (deg)	$\frac{g_1}{g_0} e^{-\Delta E/kT}$, T = 200 K	$\frac{g_1}{g_0} e^{-\Delta E/kT}$, T = 300 K	
ν_1	1	2001	4.57×10^{-5}	1.28×10^{-3}	
ν_2	2	962	0.0164	0.0813	i.e. 8%
ν_3	1	3386	4.52×10^{-8}	1.27×10^{-5}	

(5) Note that at $\bar{\nu} = 800$ cm^{-1}, where measurements are available at 3 temperatures, $\log n_s k_n$ is approximately linear in $\left(\frac{1}{T}\right)$.

4 points

So assume this relation is also valid at $\bar{\nu} = 909$ cm^{-1}.

This gives an estimate for 280 K : $n_s k_n = 3.64 \times 10^{-22}$ cm^2

$$k_n = \frac{n_s k_n}{n_s} = \frac{3.64 \times 10^{-22}}{2.687 \times 10^{19}} = 1.355 \times 10^{-41} \text{ cm}^{-5}$$

If precipitable water is 1 g cm^{-2}, spread over 2 km of height, the density is $\rho_v = 1.673 \times 10^{17}$ molecules / cm3.

$$\beta_a = \rho_v^2 \cdot k_n \qquad \tau = \beta_a \Delta z .$$

W (g cm^{-2})	ρ_v (10^{17} molecules/cm^3)	β_a (km^{-1})	τ	$e^{-1.66\tau}$	t (%)	a (%)
0.5	0.837	0.00949	0.019	0.969	97	3
1	1.673	0.0379	0.0758	0.882	88	12
2	3.346	0.152	0.304	0.604	60	40
3	5.619	0.341	0.682	0.322	32	68
5	8.365	0.948	1.896	0.043	4.3	96
7	11.711	1.86	3.72	0.002	0.2	100

Water vapor density decreases with height as does pressure. However, the error in our calculation is only about 2% if we assume w.v. density is constant with height to 2km.

3 points

6 The upward spike at 15-μm is emission from the stratosphere, warmer than the tropopause.
The upward spike at 9.6 μm is emission from the surface, warmer than the stratosphere.

(1) $V_m = \sqrt{\frac{2kT}{m}}$ $T = 273\ K$

(2 points)

molecular weights: ${}^{132}Xe = 132$

$N_2 = 28.$

$$V_m = \begin{cases} 402\ m/sec & \text{for } N_2 \\ 185\ m/sec & \text{for } Xe^{132} \end{cases}$$

(2) points

Pressure at which $\alpha_L = \alpha_D$.

$$\alpha_L = \alpha_{L0}\frac{p}{p_0}\left(\frac{T_0}{T}\right)^{1/2} \quad ; \quad \alpha_D = \frac{\nu_0}{c}\left(\frac{2kT}{m}\ln 2\right)^{1/2}.$$

Setting $\alpha_L = \alpha_D$, and solving for p/p_0:

$$\frac{p}{p_0} = \frac{1}{c\,\alpha_{L0}}\left(\frac{2k\ln 2}{T_0}\right)^{1/2} T\nu_0 m^{-1/2}$$

$T_0 = 273$; $\alpha_{L0} = 0.1\ cm^{-1}$

so $p/p_0 = 2.17\times10^{-7}\ T\nu_0 m^{-1/2}$ where $\begin{cases} T \text{ in } {}^\circ K \\ \nu_0 \text{ in } cm^{-1} \\ m \text{ in daltons} \end{cases}$

gas	m	ν_0	p (mbar) T = 200K	p (mbar) T = 300K
CO_2	44	667	4.4	6.6
H_2O	18	1600	16.6	24.8
H_2O	18	100	1.0	1.6

(3) (3 points)

Effective collision radius,

$$\alpha_L = \frac{2 r_{12}^2}{(2k\pi\mu)^{1/2}} \left(\frac{P}{T^{1/2}}\right)$$ where α_L is in Hz = sec^{-1},

and πr_{12}^2 = collision cross-section per molecule.

$$\therefore \; r_{12}^2 = (2\pi k \mu T)^{1/2} \alpha_L / 2p$$

Take μ for H_2O in N_2 : $M_1 = M_{H_2O} = 18$ g/mole
$M_2 = M_{N_2} = 28$ g/mole.

$$\mu = \frac{M_1 M_2}{M_1 + M_2} = 10.96 \text{ g/mole.}$$

$\alpha_L (cm^{-1}) = 0.1\ cm^{-1}$ at $T = 273K$ and $p = 1$ atm.

so $\alpha_L (Hz) = (0.1\ cm^{-1})(3 \times 10^{10}\ cm/sec) = 3 \times 10^9\ sec^{-1}$

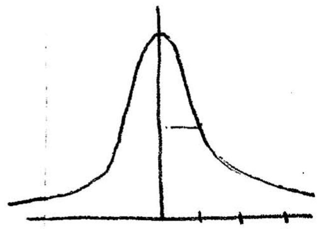
Then $(2\pi k \mu T)^{1/2} = 2.08 \times 10^{-23}$ kg · m · sec^{-1} molecule^{-1}.

And $r_{12}^2 = 3.11 \times 10^{-19} \frac{m^2}{molecule} = 31.1\ \text{Å}^2$/molecule.

$r_{12} = \sqrt{31.1} = \underline{5.6\ \text{Å}} = 0.56$ nm.

Compare this to 3.2 Å, the sum of vander Waals radii of H_2O and N_2.

(4) 3 points



Find $\dfrac{k(\nu-\nu_0=3\alpha)}{k(\nu-\nu_0=0)}$.

$\boxed{\text{Doppler}}$ $k = S\dfrac{c}{\nu_0}\left(\dfrac{m}{2\pi kT}\right)^{1/2} \exp\left[\dfrac{-mc^2}{2kT\nu_0^2}(\nu-\nu_0)^2\right]$

$$\alpha_D = \frac{\nu_0}{c}\left(\frac{2kT}{m}\ln 2\right)^{1/2}$$

so $k = \dfrac{S}{\alpha_D}\sqrt{\dfrac{\ln 2}{\pi}} \exp\left[\dfrac{-\ln 2}{\alpha_D^2}(\nu-\nu_0)^2\right]$.

Note. If α_D means halfwidth, then the $\sqrt{\ln 2}$ appears as given here. Some authors incorporate the $(\ln 2)^{1/2}$ into their "α_D".

$k(\nu_0) = \dfrac{S}{\alpha_D}\sqrt{\dfrac{\ln 2}{\pi}}$, $k(\nu-\nu_0=3\alpha_D) = \dfrac{S}{\alpha_D}\sqrt{\dfrac{\ln 2}{\pi}} \exp\left[\dfrac{-\ln 2}{\alpha_D^2}(3\alpha_D)^2\right]$

so $\dfrac{k(0)}{k(3\alpha)} = \exp(9\ln 2) = 2^9 = \underline{\underline{512}}$.

$\boxed{\text{Lorentz}}$. $k = \dfrac{S\alpha/\pi}{(\nu-\nu_0)^2+\alpha_L^2}$

$$k(\nu_0) = \frac{S\alpha_L/\pi}{\alpha_L^2} = \frac{S}{\alpha_L\pi}$$

$$k(3\alpha) = \frac{S\alpha_L/\pi}{(3\alpha_L)^2+\alpha_L^2} = \frac{S\alpha_L}{10\pi\alpha_L}$$

$$\frac{k(\nu_0)}{k(3\alpha)} = \underline{\underline{10}}.$$