ATMS 533. Problem Set 1 Tuesday 1 October 2013. Due Tuesday 8 October.

1. Relative brightness of snow, Moon, and Sun

(a) (2 points) Given that the total angular diameter of the Sun, as seen from Earth, is 32 minutes of arc, calculate the total solid angle $\Delta\omega$ occupied by the Sun in the sky, and the fraction f of the sky occupied by the Sun.

(b) (3 points) Consider a horizontal snowfield on the Earth's surface, with a flux-reflectance ("albedo") r_{snow}, illuminated by the Sun at zenith angle 60°. Assuming that the snow reflects radiation isotropically (and assuming no attenuation by the atmosphere), show that the brightness (equivalent to "intensity", or "radiance", units W $m^{-2}sr^{-1}$) of the snow, I_{snow}, relative to that of the Sun, I_{sun}, is

$$I_{snow} / I_{sun} = f \cdot r_{snow} \qquad (1)$$

(c) (5 points) Consider a full moon, full but not eclipsed. Assuming that the Moon has uniform albedo r_{moon}, and that all parts of the lunar surface reflect isotropically, the reflected brightness will vary across the illuminated hemisphere of the Moon because the Moon is a curved surface, so that different zones on the Moon are illuminated at different zenith angles. Assume that the Moon-Sun distance differs negligibly from the Earth-Sun distance, and that the lunar radius is much smaller than the Earth-Moon distance. Show that the average brightness of the full moon, I_{moon} (averaged over the lunar disk, as seen from the Earth), relative to that of the Sun, is

$$I_{moon} / I_{sun} = \frac{4}{3} f \cdot r_{moon} \qquad (2)$$

(d) (2 points) Evaluate (1) and (2) to obtain the relative brightnesses of Sun, snow, and Moon, assuming that the snow and Moon have albedos 1.0 and 0.1 respectively. Compare your value of I_{moon}/I_{sun} to the measured value 1/430,000, and give a possible reason for the difference.

2. Relative densities of molecules and photons (4 points)

(a) For an overhead sun, the downward flux of solar energy onto the top of the atmosphere is 1370 W m^{-2} (the "solar constant"). Assuming the Earth's albedo is 0.3, and that the Earth is an isotropic ("Lambertian") reflector, compute the number-density of solar photons (photons/m^3) in the upper atmosphere (i.e. above all scattering and absorbing layers), when the Sun is overhead. Use the approximation that all solar photons are at the visible wavelength λ = 500 nm (then you can just convert between energy density u, mean intensity $\bar{I}$, and photon density). Ignore the curvature of the Earth.

(b) Find the height in the atmosphere where the number-density of molecules is the same as the number-density of solar photons. Assume an isothermal atmosphere with temperature 273 K. Assume the atmospheric density ρ decreases exponentially with height z, as $\rho(z) = \rho(0) \exp(-z/H)$, with scale-height H=8 km.

ATMS 533 Homework 1 answer

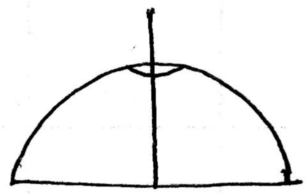
Relative brightness of snow, moon, sun:

(a) Angular diameter of sun = 32.0 minutes.

Angular radius = 16.0 min = 0.26667 deg.

Total solid angle of sun:

For convenience set up the coordinate system so that sun is centered at $\theta = 0$.



Then it's azimuthally symmetric.

$$\Delta\omega = \int_0^{2\pi} d\phi \int_0^{16\,\text{min}} \sin\theta \, d\theta$$

$$\Delta\omega = 2\pi \left[\cos 0 - \cos(16\,\text{min})\right]$$

$$= 6.805 \times 10^{-5}\ \text{sr}.$$

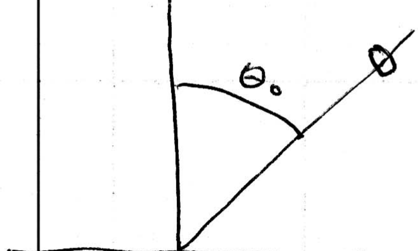
Fraction of the sky occupied by sun is $f = \frac{\Delta\omega}{2\pi} = 1.083 \times 10^{-5}$

(b) Intensity from sun is I_{sun} (average intensity across the solar disk).

Flux received from sun by a horizontal snowfield on earth's surface:

$$F\downarrow = \int I_{sun} \cos\theta \, d\omega \approx I_{sun} \cos\theta_0 \int d\omega$$

because θ doesn't vary much across the sun.



$$F\downarrow = I_{sun} \cos\theta_0 \, \Delta\omega$$

This flux is incident on the snow.

Reflected flux:

$$F\uparrow(\text{snow}) = r_{snow} F\downarrow = r_{snow} I_{sun} \cdot \Delta\omega \cdot \cos\theta_0$$

where r_{snow} = flux-reflectance ("albedo") of snow.

Assume snow is an isotropic reflector.

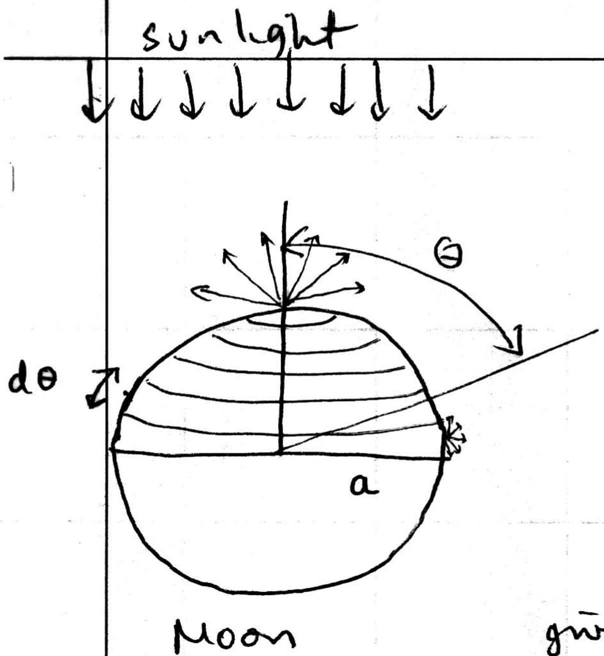
Then $$I_{snow} = \frac{F\uparrow(\text{snow})}{\pi} = \frac{r_{snow} I_{sun} \Delta\omega \cos\theta_0}{\pi}$$

and $$\frac{I_{snow}}{I_{sun}} = \frac{r_{snow} \Delta\omega \cos\theta_0}{\pi} \underset{\theta_0 = 60^\circ}{=} \frac{r_{snow} \Delta\omega}{2\pi} = f \cdot r_{snow}$$

So the brightness ratio of snow to sun is the same as the fraction of the sky occupied by the sun. (if the Sun is at an "average" zenith angle of 60°, and if $r_{snow} = 1$ (approximately true at visible wavelengths).

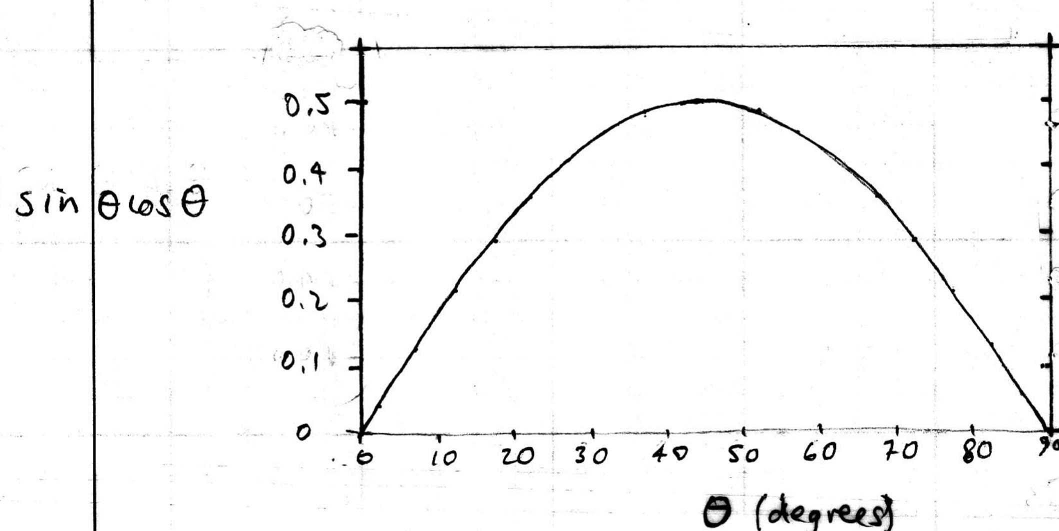
(c) Full moon.

sunlight



Area of a zone of angular width $d\theta$ is $2\pi a^2 \sin\theta \, d\theta$, where a is the moon's radius.

Projected area of this zone as seen from earth is $2\pi a^2 \sin\theta \cos\theta \, d\theta$. This function gives the relative contribution from each zone to the lunar disk seen from earth. This weighting function has maximum value at $\theta = 45°$:



Flux incident on each zone (assuming moon-sun distance is negligibly different from earth-sun distance):

$$F_{\downarrow} = I_{sun} \cos\theta \, \Delta\omega .$$

Flux reflected from each zone:

$$F_{\uparrow}(\theta) = r_{moon} I_{sun} \cos\theta \, \Delta\omega .$$

Intensity emerging from each zone, assuming isotropic reflectance:

$$I(\theta) = \frac{F_{\uparrow}(\theta)}{\pi} = r_{moon} I_{sun} \cos\theta \frac{\Delta\omega}{\pi} .$$

Average intensity of moon as seen from earth:

$$I_{moon} = \frac{\int_0^{\pi/2} I(\theta) \, W(\theta) \, d\theta}{\int_0^{\pi/2} W(\theta) \, d\theta} ,$$

where $W(\theta)$ is the weighting function for each zone $d\theta$:

$$W(\theta) = 2\pi a^2 \sin\theta \cos\theta .$$

So

$$I_{moon} = \frac{\int_0^{\pi/2} r_{moon} I_{sun} \cos\theta \frac{\Delta\omega}{\pi} 2\pi a^2 \sin\theta \cos\theta \, d\theta}{\int_0^{\pi/2} 2\pi a^2 \sin\theta \cos\theta \, d\theta}$$

$$= r_{moon} I_{sun} \frac{\Delta\omega}{\pi} \frac{\int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta}{\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta}$$

Substitute $\mu = \cos\theta$, $d\mu = -\sin\theta\, d\theta$.

So $$\frac{\int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta}{\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta} = \frac{\int_0^1 \mu^2 d\mu}{\int_0^1 \mu\, d\mu} = \frac{1/3}{1/2} = 2/3$$

So $$I_{moon} = r_{moon} I_{sun} \frac{\Delta\omega}{\pi} \cdot \frac{2}{3}$$

or $$\boxed{\frac{I_{moon}}{I_{sun}} = \frac{4}{3} \frac{r_{moon}\Delta\omega}{2\pi}} = 1.444 \times 10^{-6}$$ if $r_{moon} = 0.1$

(d) If snow has albedo 1.0 and moon has albedo 0.1

then $I_{moon} : I_{snow} : I_{sun}$

is $1 : 7.5 : 690{,}000$.

In actuality the sun is only **430,000** times as bright as the full moon. Our error is due to the assumption that moon reflects isotropically. It actually exhibits enhanced <u>backscatter</u>.

PHYSICS and ASTRONOMY of the MOON

Second Edition

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ACADEMIC PRESS New York and London • 1971

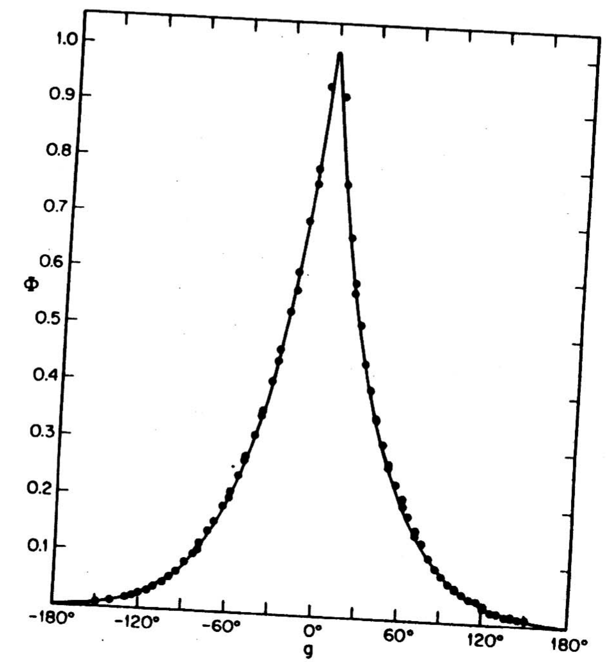


Fig. 2. Integral phase function of the Moon. Points—data of Rougier (1933); line—theoretical function of Hapke (1963).

2. *Differential Phase Function*

The radiance factor or brightness r of a portion of the surface of a planet can be expressed in the form

$$r = A\phi(i, e, g), \tag{3}$$

where r is the ratio of the brightness of the area of interest to the brightness of a Lambert screen positioned normal to the incident rays; A is the normal albedo of the area; i is the angle between the incident rays and the normal to the surface; e is the angle of observation; g is the phase angle; and ϕ is known as the photometric function of the surface. By definition, $\phi = 1$ at $g = 0$. Through the relations

$$\cos e = \cos\beta\cos\alpha \quad \text{and} \quad \cos i = \cos\beta\cos(\alpha + g), \tag{4}$$

where α and β are the luminance longitude and latitude, respectively, the radiance factor may be expressed as

$$r = A\phi(\alpha, \beta, g). \tag{5}$$

6

2 Relative densities of molecules and photons.

(a1) (i) Downward solar energy for overhead sun $Q_0 = 1370 \ \mathrm{W m^{-2}}$.

Photon energy $\epsilon = h\nu$ (Joules / photon)

Flux of downward photons $F_{down} = \dfrac{Q_0}{h\nu}$ $\left(\dfrac{\text{photons}}{\mathrm{m^2 sec}}\right)$

Number-density of downward photons

$$n_{down} = \frac{F_{down}}{c} = \frac{Q_0}{h\nu c} \quad \left(\frac{\text{photons}}{\mathrm{m^3}}\right)$$

(a2) (ii) Upward photons:

Downward solar flux onto Earth is $Q_0 \cos\theta_0$.
Assume thickness of atmosphere is small compared to radius of Earth, so the upward photons all come from nearby, where the sun is still nearly overhead.

So downward flux onto Earth is Q_0.

Upward flux is αQ_0, where α = albedo $= 0.3$.

Upward intensity $\dfrac{\alpha Q_0}{\pi}$ (isotropic in upward hemisphere)

Mean intensity of upward radiation $\bar{I}_{up} = \dfrac{\alpha Q_0}{\pi}$

Energy density of upward radiation $u_{up} = \dfrac{2\pi}{c} \bar{I}_{up}$

$$u_{up} = \frac{2\pi}{c}\frac{\alpha Q_0}{\pi} = \frac{2\alpha Q_0}{c} \quad (\mathrm{J\,m^{-3}})$$

Number-density upward $n_{up} = \dfrac{u_{up}}{h\nu} = \dfrac{2\alpha Q_0}{h\nu c}$

(a3) Total number-density of photons $n_p = n_{up} + n_{down}$

$$n_p = \frac{Q_0}{h\nu c}\left[1+2a\right] = \frac{Q_0 \lambda}{hc^2}(1+2\alpha)$$

$$= \frac{1370\ \text{J}}{\text{m}^2\ \text{sec}} \times \frac{0.5\times10^{-6}\ \text{m}}{6.63\times10^{-34}\ \text{J}\cdot\text{sec}} \times \frac{1\ \text{sec}^2}{(3\times10^8)^2\ \text{m}^2}(1+0.6) = 1.8\times10^{13}\ \frac{\text{photons}}{\text{m}^3}$$

(b) <u>Number-density of molecules</u> n_m

Ideal gas law $p = n_m kT$, where

k is Boltzmann's constant $1.38\times10^{-23}\ \text{J}\,\text{K}^{-1}\,\text{molecule}^{-1}$

n_m is number-density of molecules

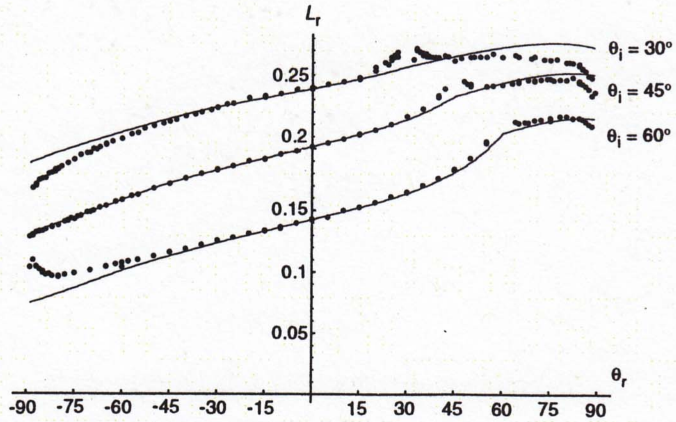
For temperature $T = 273\ \text{K}$

and pressure $p = 1$ atmosphere $= 101325$ Pa,

$n_m = 2.7\times10^{25}$ molecules/m^3 at sea level.

So at sea level $\dfrac{n_m}{n_p} = \dfrac{2.7\times10^{25}}{1.8\times10^{13}} = e^{28}$.

So $n_m = n_p$ at 28 scale heights, or $\underline{\underline{220\ \text{km}}}$



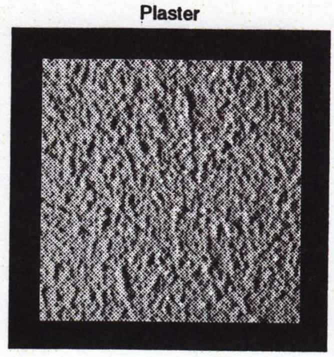
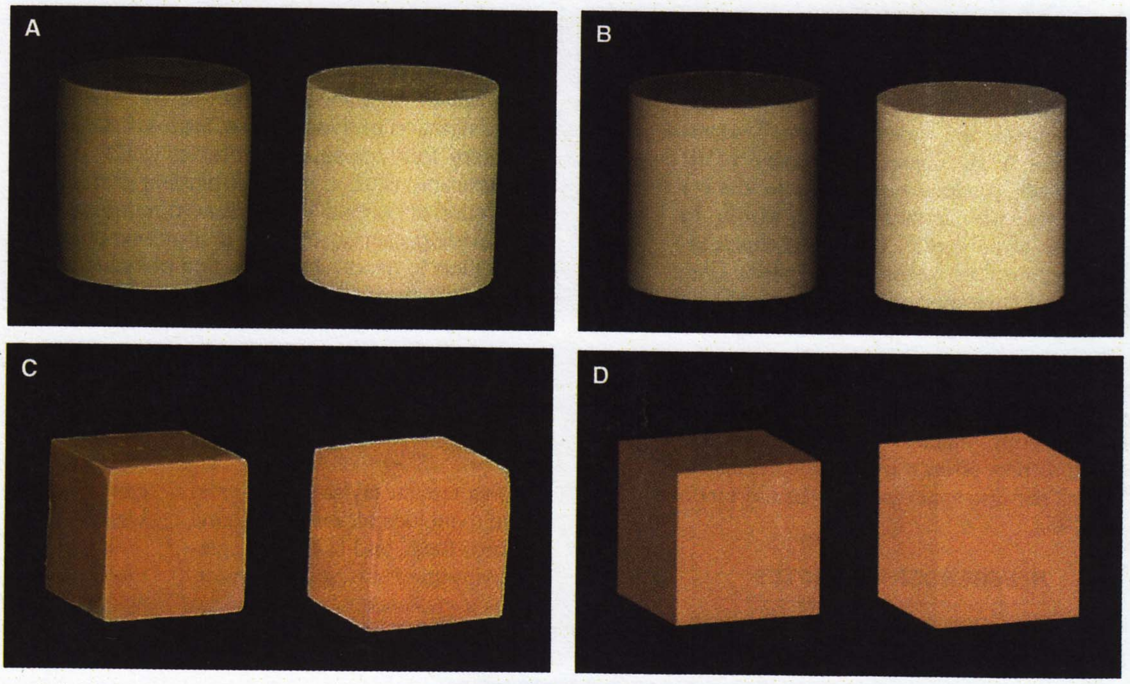


Fig. 3. Measured reflectance (dots) is compared with reflectance predicted by the model (solid lines) for plaster. Surface radiance L_r, computed as an average over the entire surface patch, is plotted as a function of sensor direction θ_r for three angles of incidence θ_i. Albedo ρ and roughness σ were selected to achieve the best fit. In these measurements, the source direction, sensor direction, and the mean surface normal are coplanar ($\phi_i = \phi_r = 0$). Surface brightness increases as the sensor approaches the source direction, violating Lambert's law that predicts brightness to be independent of viewing direction. This brightness increase is also in contrast to surface reflection mechanisms that produce peaks around the specular direction. In these and other experiments (*28*), the proposed model is found to be in strong agreement with measured data. The narrow peak observed in the source direction is attributed to the opposition effect (*30*). This phenomenon is relatively less important to visual perception as it requires the observer and the source to be within a few degrees from each other, a situation difficult to emulate in practice without either one obstructing the view of the other. The scope of the proposed model is broadened by combining it (*28*) with previously suggested ones for surface reflection (*26*) that are based on similar roughness assumptions. Validity of such a combined model was verified with samples such as sand, cloth, foam, sandpaper, and wood.

masked, shadowed, masked and shadowed, and neither masked nor shadowed. The complexity of the integral is easily seen by imagining the different masking and shadowing conditions that arise as a single V-cavity is rotated in the surface plane. We arrived at a solution to the integral by first deriving a basis function for each component of the integral, and then finding coefficients for the bases through extensive numerical simulations (*28*). The accuracy of the model was verified by matching model predictions with reflectance measurements from natural surfaces such as plaster, sand, and clay (Fig. 3). In all cases, predicted and measured data were found to be in strong agreement. A systematic increase in brightness is observed as the sensor moves toward the illuminant; this backscattering is in contrast to Lambertian behavior where brightness is constant and independent of sensor direction, and also in contrast to surface reflection where a peak in brightness is expected in the vicinity of the specular direction (*26*). For applications where simplicity is desired over high precision, approximations were made to arrive at this qualitative model:



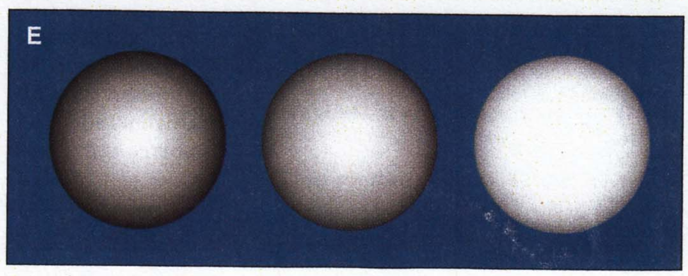


Fig. 4. (**A**) Video camera image of two cylinders made from exactly the same material (porcelain) and illuminated from approximately 10° above the camera. The right vase is much rougher than the left one, resulting in a flatter appearance. (**B**) Synthetic image of cylinders with similar dimensions, rendered with the theoretical model (left: $\sigma = 5°$, right: $\sigma = 35°$). (**C**) Camera image of two cubes made from stoneware, illuminated from approximately 18° to the left of the camera. (**D**) Synthetic image of cubes (left: $\sigma = 7°$, right: $\sigma = 40°$). In both camera and synthetic images, low macroscopic roughness of the left cube results in nearly Lambertian appearance, whereas very high roughness of the right cube results in all three faces producing almost the same brightness with clear edges no longer visible. The model and experiments suggest that for very high macroscopic roughness, when source and sensor directions are close to one another, all surface normals generate the same image brightness. Alternately, any object irrespective of its 3D shape produces just a silhouette, making it impossible to perceive shape. (**E**) Spheres illuminated and viewed from the same direction. As roughness increases (left to right: $\sigma = 0°$, $\sigma = 15°$, and $\sigma = 40°$) shading becomes flatter. For extreme roughness (far right), the sphere appears like a flat disc, as is observed in the case of a full moon.

ATMS 533. Problem Set 2 Total 15 points.
Tuesday 8 October 2013. Due Tuesday 15 October.

1. Photons of solar radiation at a particular wavelength entering an infinitely-thick cloud will travel a variety of distances into the cloud before being extinguished (scattered or absorbed). We define the "mean free path" as the average of these distances. Write an expression for the mean free path in units of optical depth. Then evaluate this expression to find the numerical value of the mean free path (in optical depth units). You may need the series expansion:

$e^x = 1 + x + x^2/2! + x^3/3! + \ldots$

Hint: To find the number of photons whose paths end in the interval $\Delta\tau$ between τ_i and $\tau_i + \Delta\tau$, compute the number of initial photons that reach τ_i and the number that reach $\tau_i + \Delta\tau$.

- - - *continued over* - - -

3 ② The phase function for Rayleigh scattering is

$$p(\Theta) = \frac{3}{4}\left(1 + \cos^2\Theta\right).$$

Show that this is properly normalized.

3 ③ For a plane-parallel geometry where the incident propagation vector $\hat{\Omega}'$ is given by (μ', ϕ') and the scattered propagation vector $\hat{\Omega}$ is given by (μ, ϕ), show that the cosine of the scattering angle is

$$\cos\Theta = \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\phi - \phi').$$

3 ④ The azimuthally-averaged phase function is

$$P(\mu, \mu') = \int_0^{2\pi} \frac{1}{2\pi} P(\mu, \phi, \mu', \phi')\, d\phi$$

Show that for Rayleigh Scattering,

$$p(\mu, \mu') = \frac{3}{8}\left(3 - \mu^2 + [3\mu^2 - 1]\mu'^2\right)$$

3 ⑤ Plot the survival probability of an emitted photon in an infinite medium as a function of the number of extinction events experienced, for

a) $\tilde{\omega} = 0.9$

b) $\tilde{\omega} = 0.999$

c) $\tilde{\omega} = 0.99999$

[Think carefully about the best way to display these results on ~~a~~ the graph.]

① Let N_0 be the total number of photons starting at optical depth zero ($\tau = 0$). The fraction of photons reaching distance (optical depth) τ_i is

$$N/N_0 = e^{-\tau_i}.$$

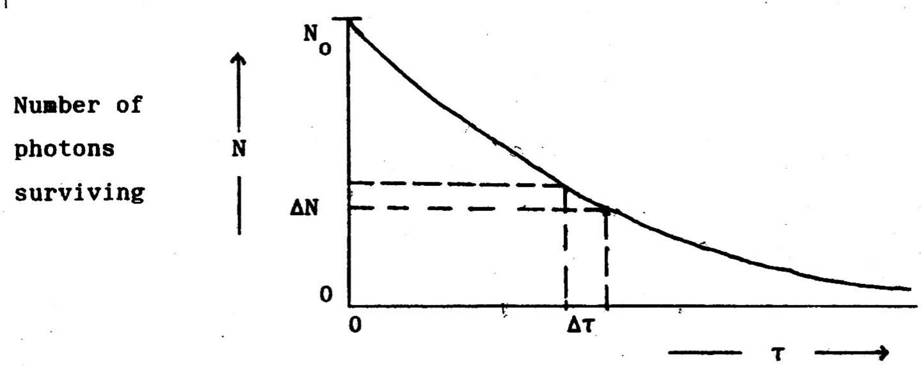
The fraction reaching distance $\tau_i + \Delta\tau$ is $e^{-(\tau_i+\Delta\tau)}$.

The fraction $\Delta N_i/N_0$ extinguished between τ_i and $\tau_i+\Delta\tau$ is therefore

$$\frac{\Delta N_i}{N_0} = e^{-\tau_i} - e^{-(\tau_i+\Delta\tau)} = e^{-\tau_i} - e^{-\tau_i}\underbrace{e^{-\Delta\tau}}_{\approx 1-\Delta\tau}$$

$$= e^{-\tau_i} - e^{-\tau_i} + \Delta\tau e^{-\tau_i}$$

$$= \Delta\tau e^{-\tau_i}$$



The mean free path is

$$\frac{\Sigma\, \Delta N_i\, \tau_i}{\Sigma\, \Delta N_i} = \frac{\Sigma\, \Delta\tau e^{-\tau_i}\, \tau_i}{\Sigma\, \Delta\tau e^{-\tau_i}} = \frac{\int \tau e^{-\tau}\, d\tau}{\int e^{-\tau}\, d\tau} = \frac{1}{1} = 1$$

Numerator: $\left[\int_0^\infty \tau e^{-\tau}\, d\tau = \tau e^{-\tau} + e^{-\tau}\right]_\infty^0 = 0 + 1 - 0 - 0$

To show that $\lim_{\tau\to\infty} (\tau e^{-\tau}) = 0$, show that $\frac{\tau}{e^\tau} \to 0$ or $\frac{e^\tau}{\tau} \to \infty$:

$$\frac{e^\tau}{\tau} = \frac{1}{\tau}\left(1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \ldots\right) = \frac{1}{\tau} + 1 + \frac{\tau}{2!} + \frac{\tau^2}{3!} + \ldots \rightarrow \infty.$$

So the mean free path is one unit of optical depth.

② Show that Rayleigh phase function is normalized.

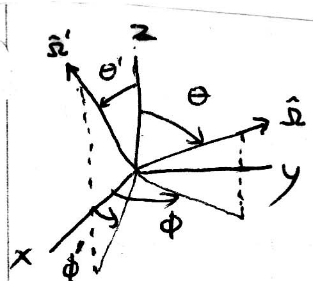
$$p(\Theta) = \frac{3}{4}(1+\cos^2\Theta)$$

$$\int \frac{p(\Theta)}{4\pi}\,d\omega = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} p(\Theta)\sin\Theta\, d\Theta$$

$$= \frac{1}{2}\cdot\frac{3}{4}\int_0^{\pi}(1+\cos^2\Theta)\sin\Theta\, d\Theta \qquad \text{Let } x \equiv \cos\Theta$$

$$= \frac{3}{8}\left[\int_{-1}^{1} dx + \int_{-1}^{1} x^2\, dx\right] = 1$$

③



x component of $\hat{\Omega}'$ is $\sin\theta'\cos\phi'$

y " " " " $\sin\theta'\sin\phi'$

z " " " " $\cos\theta'$

$$\cos\Theta = \hat{\Omega}'\cdot\hat{\Omega} = x'x + y'y + z'z$$

$$= \sin\theta'\cos\phi'\sin\theta\cos\phi + \sin\theta'\sin\phi'\sin\theta\sin\phi + \cos\theta'\cos\theta$$

$$= \sin\theta'\sin\theta\left[\cos(\phi-\phi')\right] + \cos\theta'\cos\theta$$

$\cos\theta = \mu$; $\sin\theta = \sqrt{1-\cos^2\theta} = (1-\mu^2)^{1/2}$

$$= \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\phi-\phi')$$

④ Rayleigh $p(\Theta) = \frac{3}{4}(1+\cos^2\Theta)$.

From prob. 2,

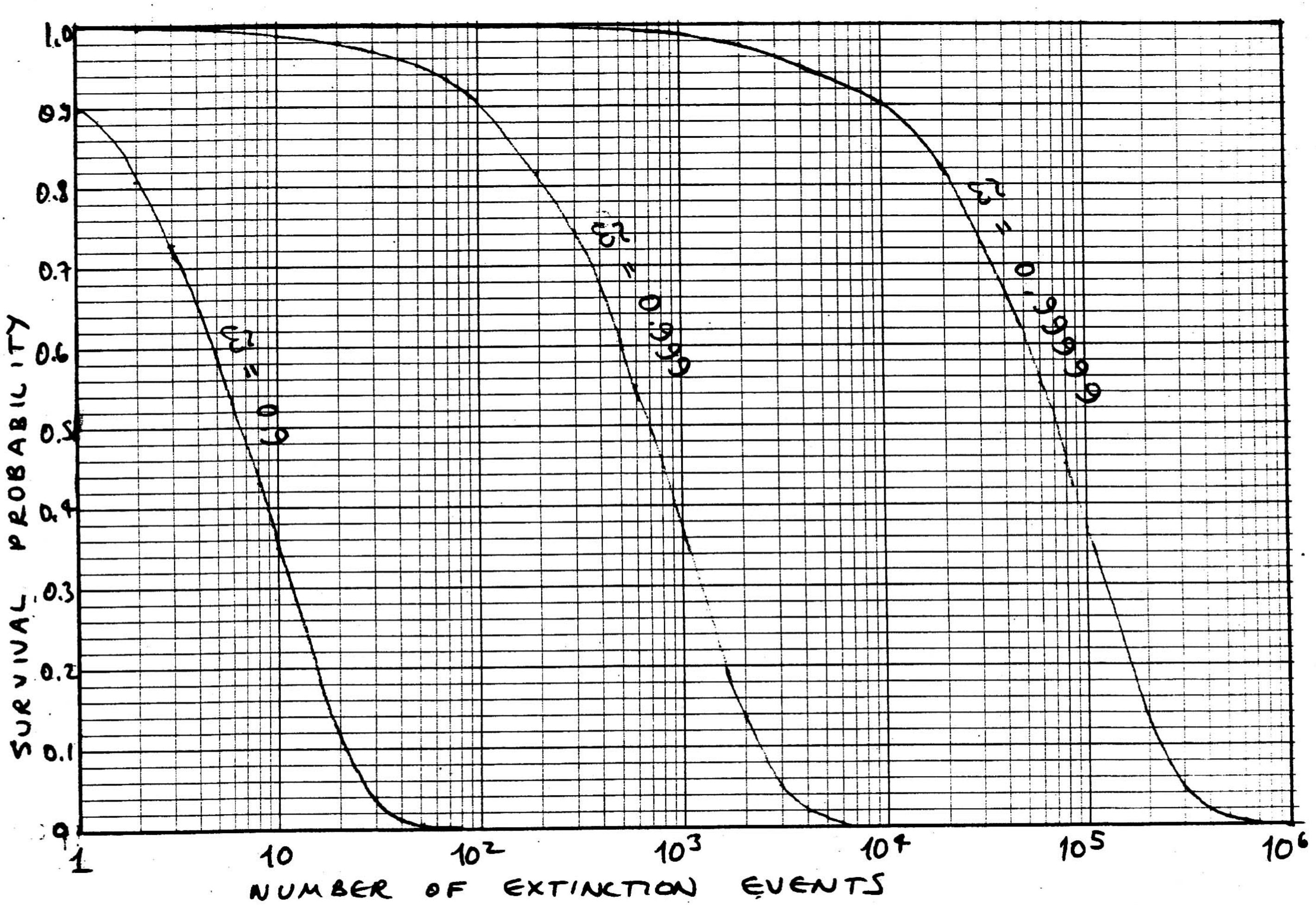
$$\frac{3}{4}(1+\cos^2\Theta) = \frac{3}{4}\Big[1+\mu^2\mu'^2 + (1-\mu^2)(1-\mu'^2)\cos^2(\phi-\phi') + \underbrace{2\mu\mu'(1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}}_{\text{call this "a"}}\cos(\phi-\phi')\Big]$$

$$p(\mu',\mu) = \int_0^{2\pi}\frac{1}{2\pi}\,p(\Theta)\,d\phi$$

$$= \frac{3}{4}\cdot\frac{1}{2\pi}\Big[2\pi(1+\mu^2\mu'^2) + (1-\mu^2)(1-\mu'^2)\underbrace{\int_0^{2\pi}\cos^2(\phi-\phi')\,d\phi}_{\underbrace{\frac{1}{2}\phi+\frac{1}{4}\sin 2\phi\Big|_0^{2\pi}}_{\pi}} + a\underbrace{\int_0^{2\pi}\cos(\phi-\phi')\,d\phi}_{\text{integral over a period }=0}\Big]$$

$$= \frac{3}{8}\left[3-\mu^2+\mu'^2(3\mu^2-1)\right]$$

⑤ Survival probability of a photon after N extinction events is $\tilde{\omega}^N$.



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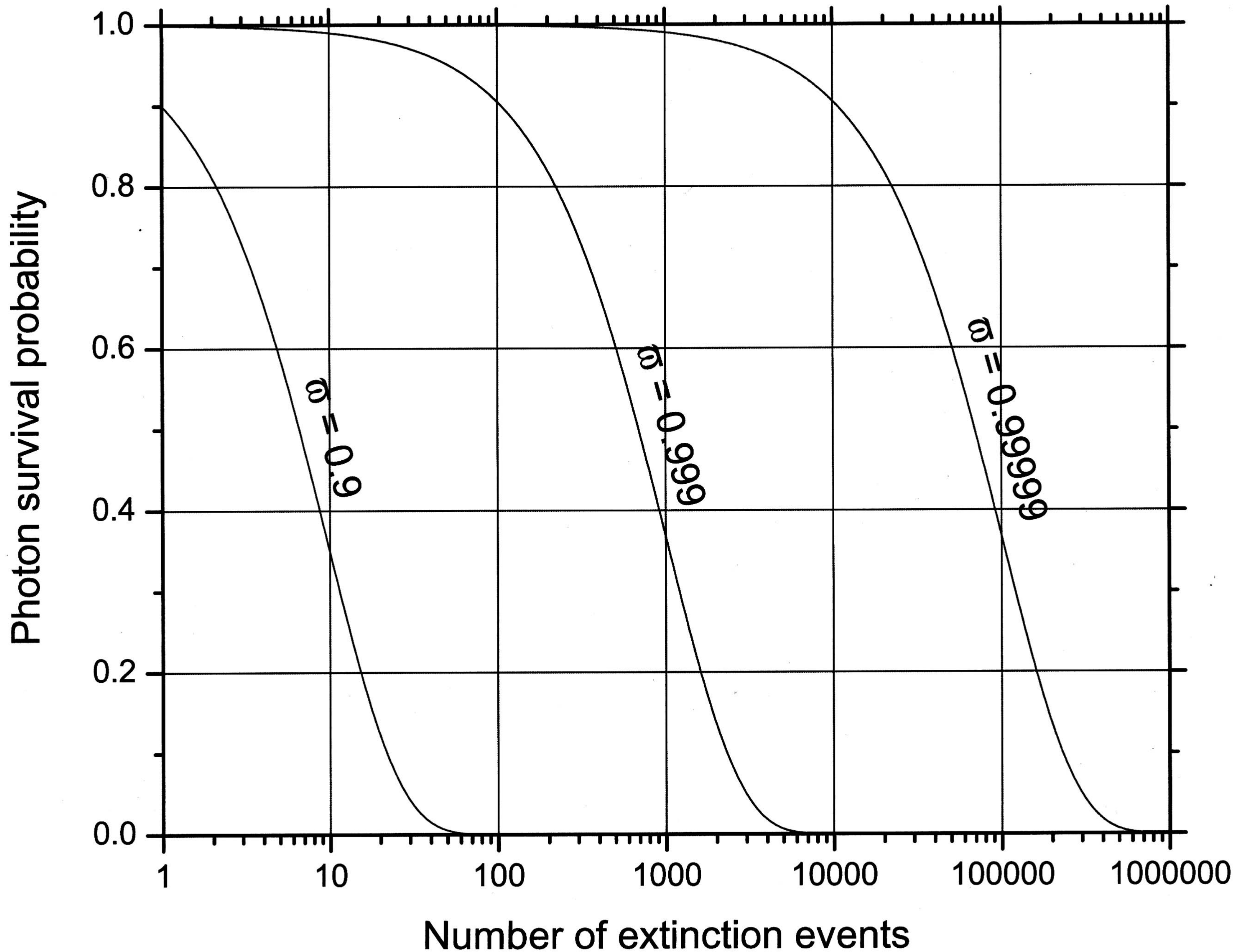


1.0
0.8
0.6
0.4
0.2
0.0
Photon survival probability
1
10
100
1000
10000
100000
1000000
Number of extinction events
ϖ = 0.9
ϖ = 0.999
ϖ = 0.99999

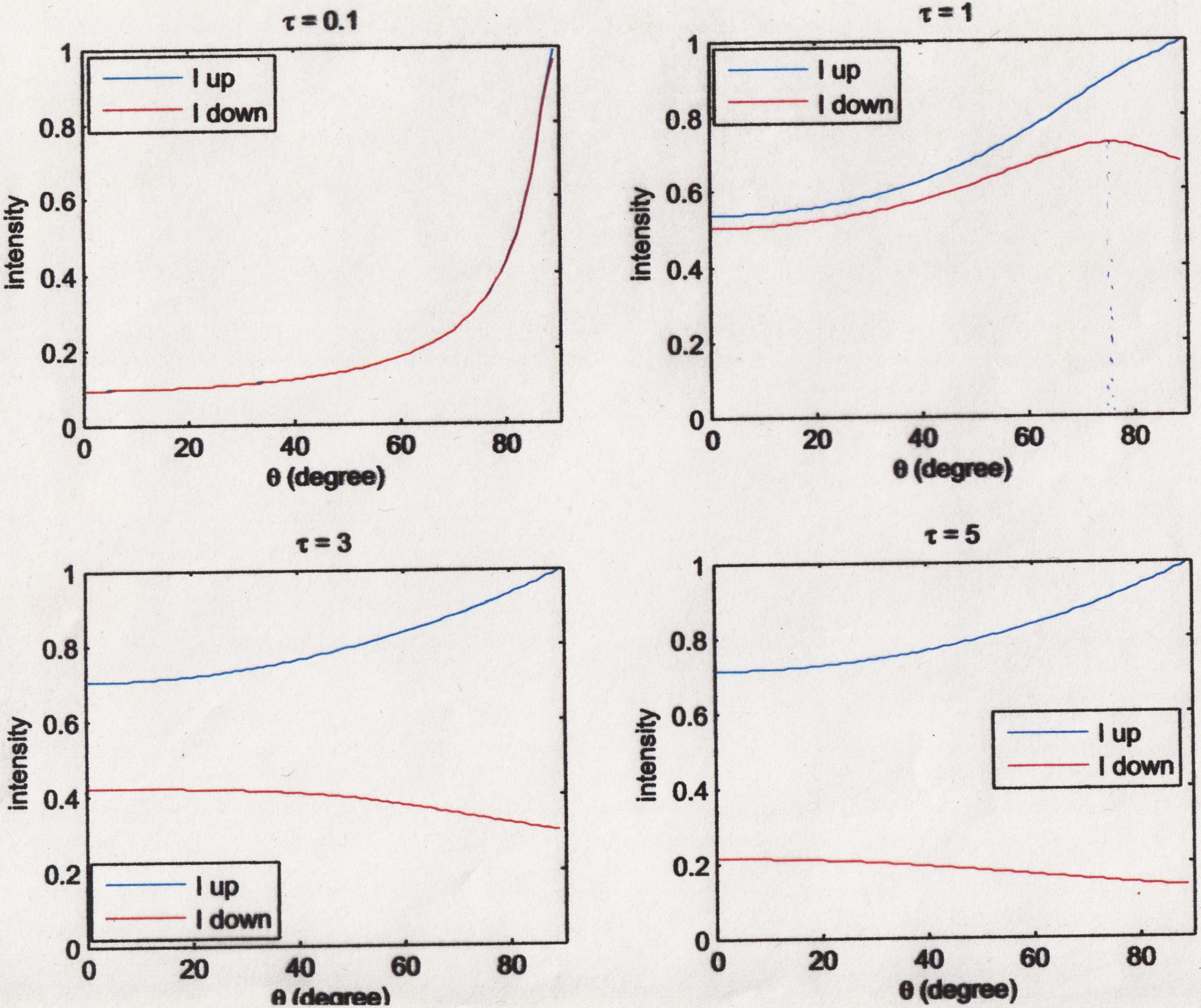
ATMS 533. Homework 3. 17 October 2013. Due Thursday 24 October.

A water-cloud of optical thickness τ^*, positioned above a black surface, is illuminated from above. Consider the mid-visible wavelength λ = 550 nm, where liquid water is non-absorbing. The cloud droplets scatter light isotropically (these are very special droplets!) The source function S is therefore independent of (θ,ϕ). To simplify the math in this problem we will assume S has the functional form $S(\tau) = \exp(-a\tau)$ for all τ, where a is a constant.

(a) Integrate the source function over τ to obtain expressions for the reflected intensity $I^+(0,\mu)$ and transmitted intensity $I^-(\tau^*,\mu)$.

(b) Plot the reflected and transmitted intensity as functions of θ from $\theta = 0°$ to $\theta = 90°$ for three cloud thicknesses: τ^* = 0.1, 1, 5, using a = 0.4. Explain the physical reasons for the shapes of the plots. If you're curious, you're of course welcome to try other values of τ^* and other values of a.

[The transmitted intensity field will contain the attenuated direct beam from the Sun as a delta-function in the solar direction (θ_o,ϕ_o), and in all other directions will consist only of scattered photons. You can omit the delta-function from your plots, and omit the incident flux at the top boundary from your derivation in part (a).]


τ = 0.1
τ = 1
τ = 3
τ = 5
I up
I down
intensity
θ (degree)
0
0.2
0.4
0.6
0.8
1
0
20
40
60
80

ATMS 533. Problem Set 4. 20 points.
Thursday 31 October 2013. Due Thursday 7 November.

Two-stream method.
Assume isotropic scattering unless otherwise indicated.

1. (6 points) Assume isotropic downward radiation at the top of a layer, with incident flux πI_o; reflectivity of the lower boundary is zero.
(a) Plot albedo *r* at the top of the atmosphere (τ=0) as a function of total optical depth τ^* from 0.01 to 100, for ϖ=1.0 (cloud at visible wavelengths) and also for ϖ=0.36 (smoke at visible wavelengths).
(b) For ϖ=1.0, plot I^+/I_o and I^-/I_o as functions of τ, for τ^*=10 and τ^*=2.
(c) For τ^*=10 and ϖ=1, plot albedo as a function of τ.

2. (5 points)
(a) For ϖ<1, show that in the optically-thin limit (τ^*<<1), the albedo at TOA is $\varpi(1-g)\tau^*$.
(b) Derive the same result for ϖ=1.
(c) Obtain this same result (albedo = $\varpi(1-g)\tau^*$) by arguing physically (not mathematically), without using a two-stream assumption.

3. (5 points) Although the two-stream approximation ignores variation of intensity with angle in order to obtain a solution to the radiative transfer equation, it is possible to obtain an estimate of the angular dependence of reflected and transmitted intensity by use of the source function (page 28 in the handout). Assume isotropic conservative scattering. Assume the incident flux at the top is πI_o (isotropic) and the reflectivity of the lower boundary is zero, as in problem 1. Obtain expressions for the intensity emerging at the top, $I^+(0,\mu)$, and the bottom, $I^-(\tau^*,\mu)$, for ϖ=1.0 and τ^*=2, by integrating the source function over τ. Plot $I^+(0)/I_o$ and $I^-(\tau^*)/I_o$ versus θ. Give a physical explanation for these results.

4. (4 points) Assume that 1.8×10^{14} grams of smoke are injected into the Earth's atmosphere and spread uniformly in a thin homogeneous layer over the entire northern hemisphere. [This is the "Nuclear Winter" scenario.] The mass extinction coefficient of the smoke is $k_e = 5.5\ m^2 g^{-1}$ at visible wavelengths ($\lambda \approx 0.5\mu m$).
(a) Find the total optical thickness τ^* of the layer.
(b) The single-scattering albedo of the smoke is ϖ=0.36. Assume that the smoke particles scatter light isotropically. Sunlight is incident on the smoke layer from above. Assuming the albedo of the Earth's surface is zero, compute the transmittance, absorptance, and reflectance of the smoke layer using the two-stream approximation. Assume isotropic incidence from above. Compare the reflectance you obtain to that given by your plot in Problem 1a.

P.S. 4 answers ATMS 533

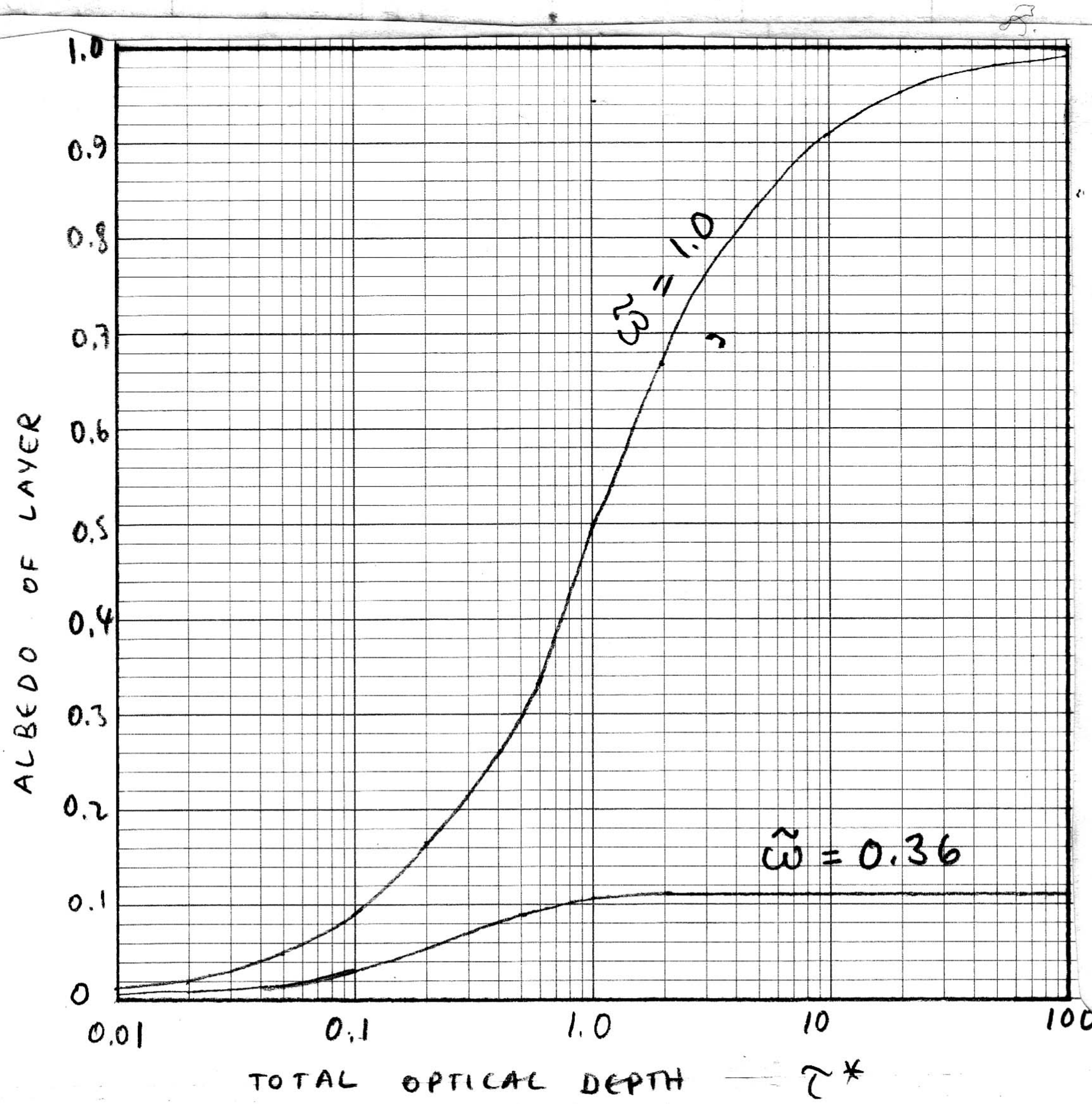
TWO-STREAM METHOD

① (a) $\tilde{\omega} = 1.0$ albedo $= \dfrac{\tau^*}{1+\tau^*}$

$\tilde{\omega} = 0.36$ (smoke) $\rho_\infty = \dfrac{1-\sqrt{1-\tilde{\omega}}}{1+\sqrt{1-\tilde{\omega}}} = 0.111$ $I^-(0) = I_0$

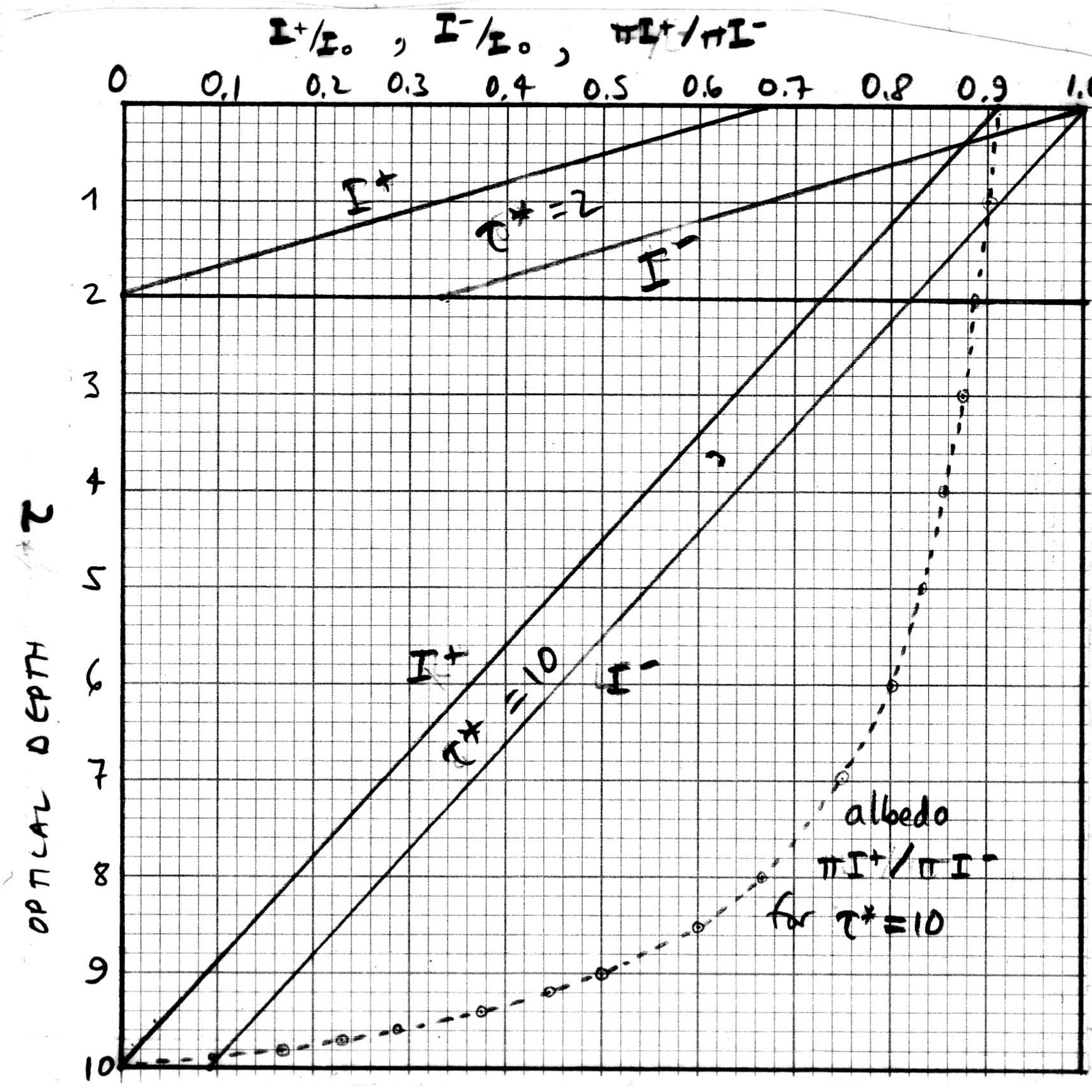
$$\text{Albedo} = \frac{\pi I^+(0)}{\pi I^-(0)} = \frac{\rho_\infty \left[e^{\Gamma\tau^*} - e^{-\Gamma\tau^*}\right]}{e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}} \quad ; \; \Gamma = 2\sqrt{1-\tilde{\omega}}$$

$$\text{Albedo} = \frac{0.111\left(e^{1.6\tau^*} - e^{-1.6\tau^*}\right)}{e^{1.6\tau^*} - 0.01235\, e^{-1.6\tau^*}}$$



(1b) $\varpi = 1.0 \quad \frac{I^+}{I_0} = \frac{\tau^* - \tau}{1+\tau^*} \qquad \frac{I^-}{I_0} = \frac{1+\tau^*-\tau}{1+\tau^*}$

(1c) Albedo at any level τ is $\pi I^+(\tau) / \pi I^-(\tau) = \frac{\tau^*-\tau}{1+\tau^*-\tau}$



(2a) Eqs. 33 & 34 give albedo for the case $\tilde{\omega} \neq 1$, $\rho_\infty \neq 1$.

For $\tau, \tau^* \ll 1$, $e^{\Gamma(\tau^*-\tau)} \approx 1 + \Gamma(\tau^* - \tau)$ etc.

$$\text{albedo} = r = \frac{\pi I^+(0)}{\pi I^-(0)} = \rho_\infty \frac{1 + \Gamma\tau^* - 1 + \Gamma\tau^*}{1 + \Gamma\tau^* + \rho_\infty^2 \Gamma\tau^* - \rho_\infty^2}$$

$$= \rho_\infty \frac{2\Gamma\tau^*}{(1+\Gamma\tau^*) - \rho_\infty^2 (1 - \Gamma\tau^*)} .$$

For $\Gamma\tau^* \ll 1$, neglect $\Gamma\tau^*$ in the denominator.

$$r \to \frac{2\Gamma\tau^* \rho_\infty}{1 - \rho_\infty^2}$$

but $$1 - \rho_\infty^2 = \frac{2\Gamma}{(\sqrt{1-\omega g} + \sqrt{1-\omega})^2}$$

so $$r = \tau^* \left[\sqrt{1-\omega g} - \sqrt{1-\omega}\right]\left[\sqrt{1-\omega g} + \sqrt{1-\omega}\right] = \tau^* \left[(1-\omega g) - (1-\omega)\right]$$

$$\boxed{r = \tau^* \tilde{\omega} (1-g) .}$$

(b) For $\tilde{\omega} = 1$ use Eq. 41:

$$\frac{\pi I^+(0)}{\pi I^-(0)} = \frac{(1-g)\tau^*}{1 + (1-g)\tau^*} \longrightarrow (1-g)\tau^* \quad \text{for } \tau^* \ll 1 .$$

(c) Physical argument:

In the optically-thin limit there is only single-scattering.

Transmittance of direct beam $t = e^{-\tau^*/\bar{\mu}} = 1 - \tau^*/\bar{\mu}$

extinction $= 1 - t = \tau^*/\bar{\mu}$

scattering $= \tilde{\omega}\tau^*/\bar{\mu}$. Fraction b is scattered back

$b = \frac{1-g}{2}$ so albedo is $\dfrac{\tilde{\omega}\tau^*}{\bar{\mu}} \dfrac{(1-g)}{2} = \boxed{\tilde{\omega}(1-g)\tau^*}$ if $\bar{\mu} = \frac{1}{2}$.

(3) Use two-stream source function to estimate the zenith-angle dependence of emergent and transmitted radiation, for $\tilde{\omega}=1$, $g=0$, $\tau^*=2$

$$I^+(0,\mu) = \int_0^{\tau^*} \frac{d\tau}{\mu} S(\tau,\mu)\, e^{-\tau/\mu}$$

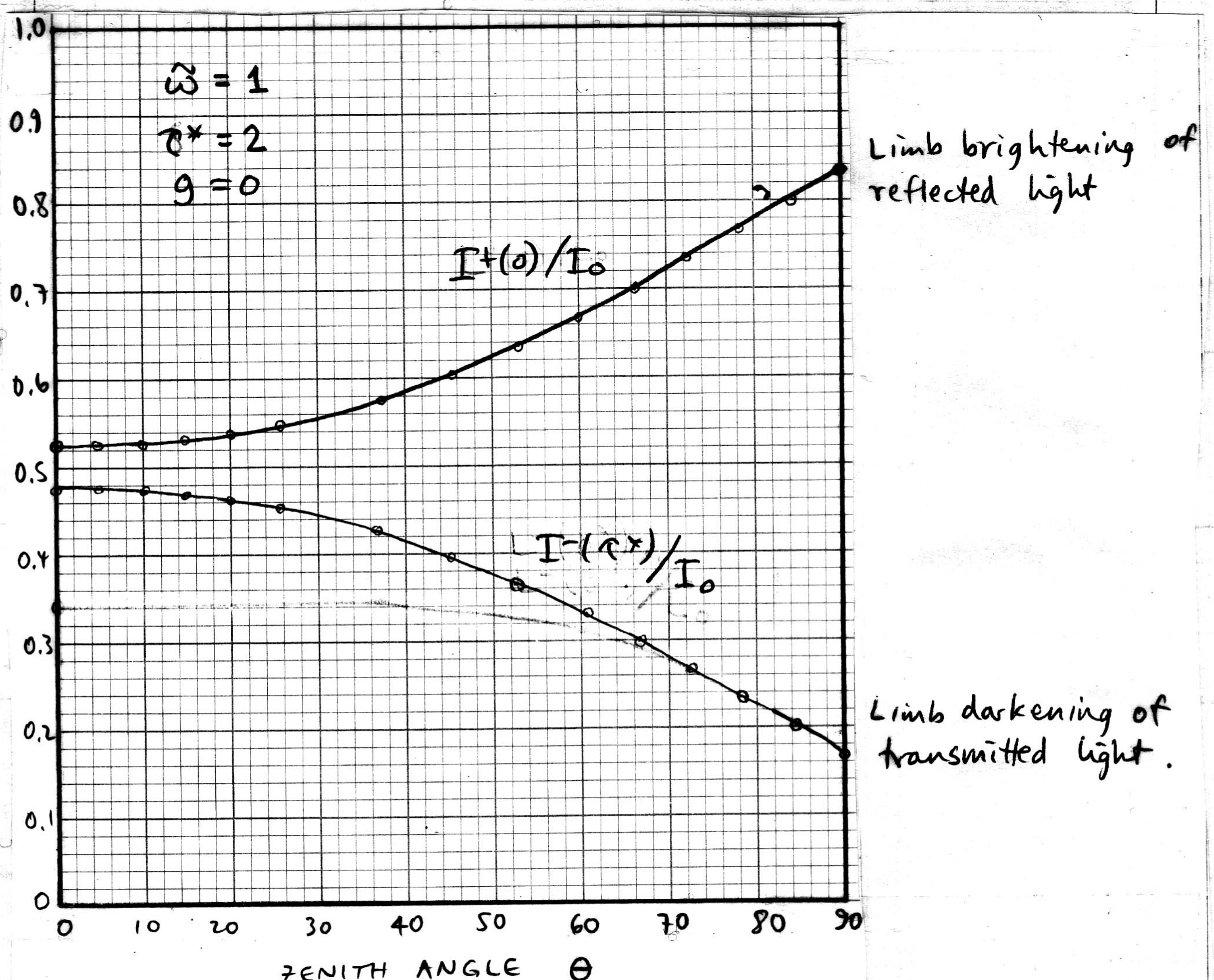
$$I^-(\tau^*,\mu) = \int_0^{\tau^*} \frac{d\tau}{\mu} S(\tau,\mu)\, e^{-(\tau^*-\tau)/\mu} + I^-(0,\mu)\, e^{-\tau^*/\mu}$$

For $\tilde{\omega}=1$, use $\dfrac{S(\tau)}{I_0} = \dfrac{1+2(\tau^*-\tau)}{2(1+\tau^*)}$ (because S is mean-intensity; $S = \dfrac{I^+ + I^-}{2}$)

Doing the integrations,

$$\frac{I^+(0,\mu)}{I_0} = \frac{1}{\tau^*+1}\left[\tau^* - \left(\mu-\tfrac{1}{2}\right)\left(1-e^{-\tau^*/\mu}\right)\right] \overset{\tau^*=2}{=} \frac{1}{3}\left[2-\left(\mu-\tfrac{1}{2}\right)\left(1-e^{-2/\mu}\right)\right]$$

$$\frac{I^-(\tau^*,\mu)}{I_0} = \frac{1}{\tau^*+1}\left[1 + \left(\mu-\tfrac{1}{2}\right)\left(1-e^{-\tau^*/\mu}\right)\right] = \frac{1}{3}\left[1+\left(\mu-\tfrac{1}{2}\right)\left(1-e^{-2/\mu}\right)\right]$$



4 "Nuclear winter" smoke cloud

(a) 1.8×10^{14} g smoke is spread over the Northern Hemisphere area which is 2.55×10^{14} m^2.

So the smoke layer thickness $\int \rho\, dz = \frac{1.8}{2.55} = 0.71 \; g\, m^{-2}$.

Optical thickness $\tau^* = \int k_{ext}\, \rho\, dz = k_{ext} \int \rho\, dz$

$$= 5.5 \frac{m^2}{g} \times 0.71 \frac{g}{m^2} = \boxed{3.9 = \tau^*}$$

(b) Use 2-stream method with $\tilde{\omega} = 0.36$

so $\Gamma = 2\sqrt{1-\tilde{\omega}} = 1.60$; $\rho_\infty = \frac{2-\Gamma}{2+\Gamma} = 0.111$; $\Gamma\tau^* = 6.26$

$$\frac{I^+(0)}{I_0} = \frac{\rho_\infty}{e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}} \left[e^{\Gamma\tau^*} - e^{-\Gamma\tau^*}\right] = 0.111 = \text{reflectivity} = r$$

$$\frac{I^-(\tau^*)}{I_0} = \frac{1-\rho_\infty^2}{e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}} = 0.002 = \text{transmissivity} = t$$

absorptance $a = 1 - r - t = 0.887$

$$\boxed{\begin{array}{l} r = 0.111 \\ a = 0.887 \\ t = 0.002 \end{array}}$$

i.e. 0.2% of the incident sunlight is transmitted through the smoke cloud.

ATMS 533. Problem Set 5. Total 12 points.
Thursday 7 November 2013. Due Thursday 14 November.

Enhancement of absorption due to scattering.

Introduction:

Use two-stream results for diffuse incidence to compute the solar heating rates for both clear and cloudy atmospheres, as follows. The incident solar flux at TOA is $Q_o\cos\theta_o$, where the solar constant is Q_o=1370 W m^{-2} and θ_o=60°. The surface albedo is zero.

Clear atmosphere. Assume that the Earth's atmosphere contains no scatterers or absorbers above 2 km height. Between z=0 and z=2 km there is still no scattering, but there is water vapor of uniform density. The total precipitable water vapor is 3 g cm^{-2}. The broadband absorption coefficient of water vapor for solar radiation is k_a=0.042 cm^2g^{-1}.

Cloudy atmosphere. The cloudy atmosphere is identical to the clear atmosphere except for the addition of a 1-km-thick cloud between z_B and z_T, where z_B=1 km and z_T=2 km. The cloud has scattering coefficient σ_s=10 km^{-1} and asymmetry factor g=0.85. The cloud droplets are non-absorbing. The water vapor content is the same as in the clear atmosphere.

Problem:

1. (4 points) Obtain an expression for the divergence of net flux (dF_{net}/dz) in the two-stream approximation
(a) for the case of pure absorption, ϖ=0;
(b) for the general case, $0<\varpi<1$.

2. (2 points) What is the single-scattering albedo
(a) in the clear air
(b) in the cloud

3. (4 points) Calculate dF_{net}/dz (W m^{-3}) as a function of height for both clear and cloudy atmospheres. Convert to an approximate heating rate by approximating the atmospheric density as constant in the lowest 2 km, ρ=1.135 kg m^{-3}, and using the heat capacity of air as 1004 J $kg^{-1}deg^{-1}$. Plot the instantaneous heating rate dT/dt in degrees per day, versus height for the clear and cloudy atmospheres.

4. (2 points) Give a physical explanation for the differences in the heating rate profiles.

Problem Set 5 answer ATMS 533

① Use p. 81 of notes.

(a) pure absorption. $\tilde{\omega} = 0$ $\Gamma = 2$ $\rho_\infty = 0$.

$$F_{net} = \pi (I^+ - I^-) = -\pi I_0 e^{-2\tau}$$

$$\frac{dF_{net}}{d\tau} = 2\pi I_0 e^{-2\tau} \quad . \quad \text{Identify} \begin{cases} \tau = \sigma_a (z_T - z) \\ d\tau = -\sigma_a dz \\ \pi I_0 = \mu_0 Q_0 \end{cases}$$

$$\frac{dF_{net}}{dz} = -2\pi \sigma_a I_0 e^{-2\tau} = \underline{\underline{-2\sigma_a \mu_0 Q_0 e^{-2\sigma_a (z_T - z)}}}$$

(b) In general:

$$F_{net} = \pi (I^+ - I^-) = \frac{-\pi I_0 (1-\rho_\infty)}{e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}} \left[e^{\Gamma(\tau^* - \tau)} + \rho_\infty e^{-\Gamma(\tau^* - \tau)} \right]$$

$$\frac{dF_{net}}{dz} = -\sigma \frac{dF_{net}}{d\tau} = \frac{-\sigma \Gamma \mu_0 Q_0 (1-\rho_\infty)}{e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}} \left[e^{\Gamma(\tau^* - \tau)} - \rho_\infty e^{-\Gamma(\tau^* - \tau)} \right]$$

where $\tau^* = \sigma (z_T - z_B)$

$\tau = \sigma (z_T - z)$

② $\sigma_a = k_a \rho = 0.042 \frac{cm^2}{g} \cdot \frac{3g}{cm^2} \cdot \frac{1}{2 km} = 0.063 km^{-1}$

[k_a obtained from Yamanouchi & Tanaka 1985: JQSRT 34 472]

In cloud, $\sigma_s = 10 km^{-1}$; $\sigma = \sigma_a + \sigma_s = 10.063 km^{-1}$

$\tau = 0$ ———————————— $z_T = 2\ \text{km}$

$\sigma_a = 0.063\ \text{km}^{-1}$

$\sigma_s = 10\ \text{km}^{-1}$

———————————— $z_B = 1\ \text{km}$

$\sigma_a = 0.063\ \text{km}^{-1}$

$\sigma_s = 0$

$\tau = \tau^*$ ———————————— $z = 0$

(a) clear air, pure absorption. $\bar{\omega} = 0$

(b) In cloud, $\bar{\omega} = \dfrac{\sigma_s}{\sigma_a + \sigma_s} = 0.99374.$

(3) <u>In cloud</u>

$g = 0.85$, $1 - \bar{\omega} g = 0.15532$, $1 - \bar{\omega} = 0.00626$, $\Gamma = 0.06236$,

$\rho_\infty = 0.70102$, $\tau^* = \sigma (z_T - z_B) = 10.063.$

<u>clear</u>

need only $\sigma_a = 0.063\ \text{km}^{-1}$

<u>cloudy, under cloud</u>:

Use clear-sky formula, but replace $\mu_0 Q_0$ by

$$\pi I^-(z = 1\ \text{km}) = \frac{\mu_0 Q_0}{1.6106}[1 - 0.49143] = 216\ \text{Wm}^{-2}$$

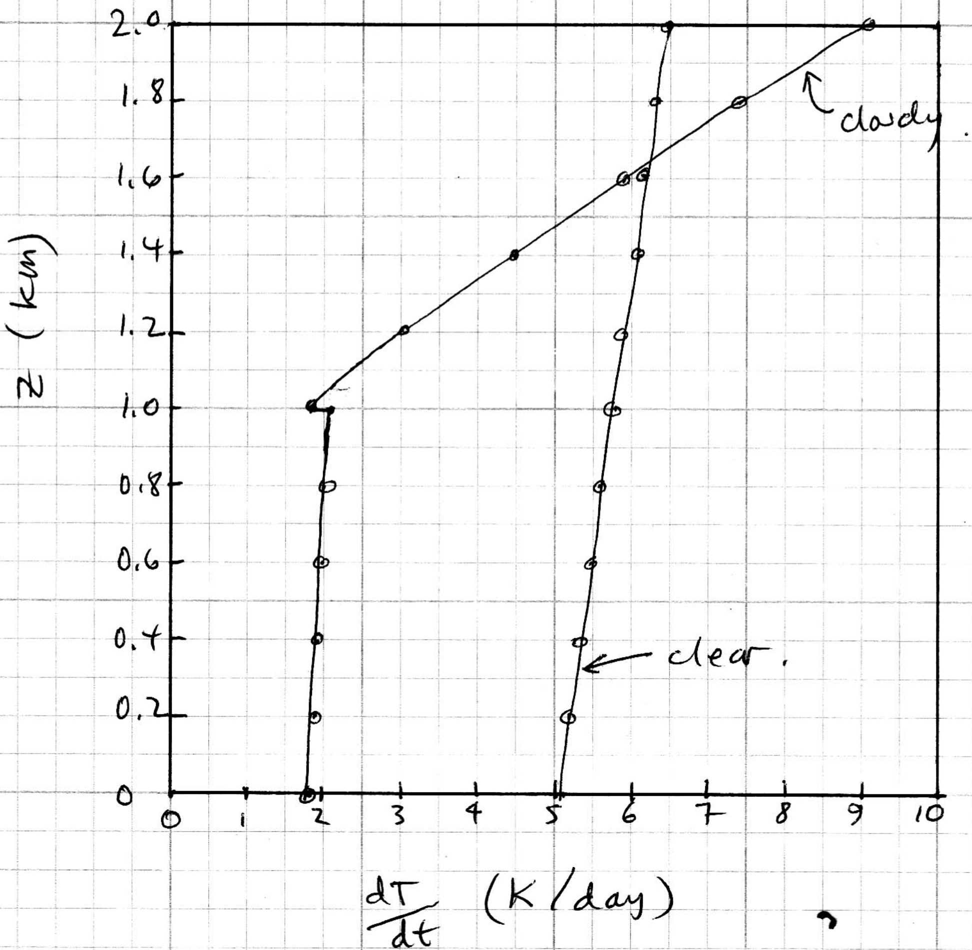
<u>conversion to dT/dt</u>

$$\frac{dT}{dt} = \frac{-1}{\rho C_p} \frac{dF_{net}}{dz}. \quad C_p = 1004 \frac{\text{J}}{\text{K} \cdot \text{deg}} \quad \rho = 1.135 \frac{\text{kg}}{\text{m}^3}$$

-------- **CLEAR SKY**--------

---------------------------**CLOUDY SKY**--

z (km)	dF_{net}/dz clear ($mW\ m^{-3}$)	dT/dt clear (K/day)	dF_{net}/dz in cloud ($mW\ m^{-3}$)	dF_{net}/dz under cloud ($mW\ m^{-3}$)	dT/dt in cloud (K/day)	dT/dt under cloud (K/day)
2.0	-86	6.5	-120		9.1	
1.8	-84	6.4	-98		7.4	
1.6	-82	6.2	-78		5.9	
1.4	-80	6.1	-59		4.5	
1.2	-78	5.9	-41		3.1	
1.0	-76	5.8	-24	-27.2	1.8	2.06
0.8	-74	5.6		-26.5		2.01
0.6	-72	5.5		-25.9		1.96
0.4	-70	5.3		-25.2		1.91
0.2	-69	5.2		-24.6		1.87
0.0	-67	5.1		-24.0		1.82



④ Clear: heating rate varies little with z because τ^* is small.

Cloudy: photon energy density is higher at cloud top because of scattering, enhances absorption over clear sky. Deeper in cloud, illumination is reduced because most photons have been scattered out the top of the cloud.

Although absorption is enhanced at the top, the total atmospheric absorption is less in the cloudy atmosphere.

ATMS 533. Problem Set 6.
Thursday 14 November 2013. Due Monday 25 November.

Discrete Ordinates Method.

You will use the DISORT (DIScrete Ordinates Radiative Transfer) program to compute intensities in a water-cloud at wavelength 1.6 μm, over a black surface. The DISORT program is a package which is not to be modified. It is called by a driver program written by Mike Town and maintained by Maria Zatko, which you can modify. The driver program specifies the inputs to DISORT.

The droplets are 10 μm in radius. A Mie-scattering calculation resulted in single-scattering albedo ϖ=0.9934 and gave 48 phase-function moments, which are listed in the driver program. [The first moment is the asymmetry parameter, g=0.85.] Select the delta-M option.

For the standard problem, use a cloud optical depth τ^*=20, with the Sun at θ_o=60°, ϕ_o=0°. Compute upward intensity at cloud top, and downward intensity at cloud bottom vs. outgoing zenith angle θ in the principal plane (the plane containing the surface normal and the Sun), in both forward and backward directions. Also compute the azimuthally averaged intensity as a function of θ (or μ) at the top, bottom, and middle of the cloud. Run the program several times: for 4, 8, 16, and 32 streams, and plot the results from all runs on the same graph. The accuracy will improve as the number of streams increases.

Repeat for solar zenith angle θ_o=78° (μ_o=0.2). Discuss the results.

Here are Maria's instructions:

You can find everything you need for the DISORT Homework assignment here:
/home/disk/p/mzatko/ATMS533_DISORT

Mike Town (one of Steve Warren's former grad students) wrote a driver that runs the DISORT radiative transfer code. This way, DISORT is not altered during this assignment. The driver (disortDriver.m) makes a tempInput.txt file that contains the variables you will specify as the driver runs. The driver passes the tempInput.txt file to the atest.out executable. An ouput text file is then created (e.g. output4_20_09934_02.txt). The executable has been designed specifically for this homework assignment. For your own reference, the DISORT code (code.F) has been included in this folder. A helpful README written by Mike is also in the ATMS533_Disort folder.

The driver must be run on Linux because the atest.out executable was piled on a Linux machine. I successfully ran the driver using both the 2007 and 2013 versions of Matlab.

Let me know if you have any questions!
Maria <mzatko@uw.edu>

ATMS 533. Problem Set 7. 14 points total
Monday 25 November 2013. Due Monday 2 December.

1. Cloud albedo at visible wavelengths: dependence on number of cloud-condensation nuclei. Use the two-stream approximation for albedo R of a conservative-scattering layer for diffuse incidence (to mimic behavior for a solar zenith angle of 60°),

$$R = [(1-g)\tau^*] / [1 + (1-g)\tau^*],$$

and the dependence of optical thickness on effective radius r and liquid water path W,

$$\tau^* = 3W / 2r\rho,$$

where we have assumed $Q_{ext}=2$. For the following derivation assume a monodispersion, i.e. all droplets have the same radius r. Assume that each cloud-condensation nucleus (CCN) grows to becomes a cloud droplet. Let the number-density N of CCN vary while liquid water path is held constant. Assume liquid water content is independent of height in the cloud.

(a) (4 points) Derive the sensitivity of albedo to fractional change in N; i.e. show that

$$dR/d\ln N = R(1-R) / 3.$$

(b) (2 points) For a 10% increase in cloud-condensation nuclei, plot the change in the cloud albedo ΔR as a function of its initial albedo R.

2. (2 points) A stratus cloud, 1 km thick, consists only of water droplets all the same size, with radius r = 10 μm. The liquid water content of the cloud is L = 0.1 g m^{-3}. The droplets have $Q_{ext}=2$, single-scattering coalbedo $(1-\varpi)=10^{-4}$ (typical of near-IR), and asymmetry factor g=0.85. Compute the total optical thickness of the cloud τ^*, and the delta-Eddington scaled optical thickness $\tau^*_{\delta E}$.

3. (2 points) A snowpack, only 1 cm thick, of density ρ=0.4 g cm^{-3}, consists only of ice spheres all the same size, with r = 100 μm. The ice spheres have $Q_{ext}=2$; $(1-\varpi)=10^{-3}$ (typical of near-IR) and g=0.89. Compute τ^* and $\tau^*_{\delta E}$.

4. (4 points) Clouds and snow contain not only particles but also water vapor, which absorbs but does not scatter radiation. Assume the relative humidity is 100% in the cloud and in the snow, both of which are at T=0°C. Assume the mass absorption coefficient of pure water vapor is 0.5 cm^2g^{-1} in the near-IR. **(a)** Compute τ^*, ϖ, and $\tau^*_{\delta E}$ for the stratus cloud in problem 2, including water-vapor absorption.

(b) For the snowpack in problem 3, compute τ^*, ϖ, and $\tau^*_{\delta E}$, including water-vapor absorption.

① ⓐ

2-stream albedo $R = \frac{(1-g)\tau^*}{1+(1-g)\tau^*} = \frac{\tau^*}{\frac{1}{1-g} + \tau^*}$

$$1 - R = \frac{1}{1+(1-g)\tau^*} = \frac{\frac{1}{1-g}}{\frac{1}{1-g} + \tau^*}$$

Let $a \equiv \frac{1}{1-g}$.

Then $R = \frac{\tau^*}{a+\tau^*}$ and $1-R = \frac{a}{a+\tau^*}$

$$\frac{dR}{d\tau^*} = \frac{1}{a+\tau^*} - \frac{\tau^*}{(a+\tau^*)^2} = \frac{a}{(a+\tau^*)^2}$$

$$\frac{dR}{d\ln\tau^*} = \tau^*\frac{dR}{d\tau^*} = \frac{\tau^* a}{(a+\tau^*)^2} = \left(\frac{a}{a+\tau^*}\right)\left(\frac{\tau^*}{a+\tau^*}\right) = R(1-R) \quad (1)$$

Liquid water content is L ; Liquid water path is W

$W = \int L(z)\,dz = L\Delta z$ assuming L independent of z.

$L = \frac{4}{3}\pi r^3 N\rho$ so $r \propto N^{-1/3}$ for constant W (constant L)

$\tau^* = \frac{3}{2}\frac{W}{r\rho}$ so $\tau^* \propto N^{1/3}$ for constant W

$\ln\tau^* = k + \frac{1}{3}\ln N$ where k is a constant.

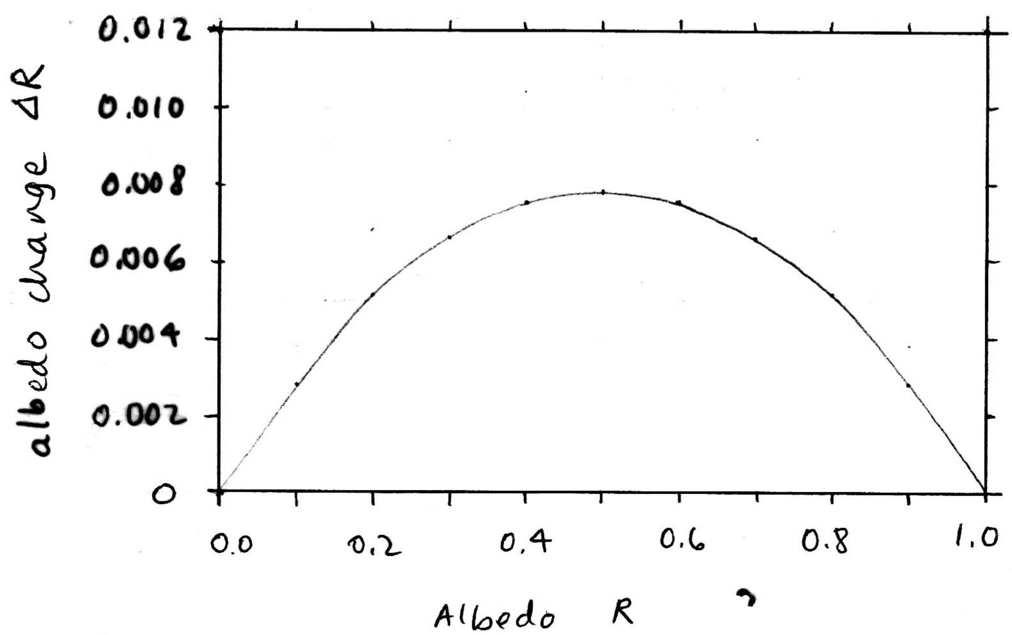
$$\frac{d\ln\tau^*}{d\ln N} = \frac{1}{3}.$$

So $$\frac{dR}{d\ln N} = \frac{1}{3}\frac{dR}{d\ln\tau^*} = \frac{R(1-R)}{3}$$

(1b) For 10% increase in N:

$$\Delta \ln N = \ln N' - \ln N = \ln\left(\frac{N'}{N}\right) = \ln 1.1 = 0.0953$$

$$\Delta R \approx \frac{R(1-R)}{3} \Delta \ln N = R(1-R)\frac{(0.0953)}{3}$$



(2) $$\tau^* = \frac{3}{4}\frac{W Q_{ext}}{r\rho} = \frac{3}{4}\frac{Q_{ext}}{r\rho}\int L\,dz = \frac{3}{4}\frac{Q_{ext}L}{r\rho}\Delta z$$

if L constant

$$\tau^* = \sigma \Delta z \quad ; \quad \sigma = \frac{3}{4}\frac{Q_{ext}L}{r\rho}$$

Using $Q_{ext} = 2$, $L = 0.1\ g\,m^{-3}$, $\rho = 1\ g\,cm^{-3}$, $r = 10\,\mu m$,

$\Delta z = 1000\ m$, we get $\boxed{\tau^* = 15.0}$

δ-Eddington scaled optical depth: $\hat{\omega} = 0.9999$, $g = 0.85$;

$$\tau^*_{\delta E} = (1 - \hat{\omega} g^2)\, \tau^* = (0.28)(15.) = \boxed{4.16 = \tau^*_{\delta E}}$$

(3) 1-cm snowpack (i.e. a thin snowpack), $L = 0.4$ g cm^{-3},

$r = 100\ \mu m = 0.01$ cm ; $Q_{ext} = 2$; $\rho_{ice} = 0.92$ g cm^{-3};

so $$\sigma = \frac{3(0.4)(2)}{4(0.92)(0.01)} = 65.2 \text{ cm}^{-1}$$

Total optical depth $\tau^* = (65.2 \text{ cm}^{-1})(1 \text{ cm}) = \boxed{65.2 = \tau^*}$

$\hat{\omega} = 0.999$, $g = 0.89$; $\tau^*_{\delta E} = 0.21\, \tau^* = \boxed{13.6 = \tau^*_{\delta E}}$

(4) Now add water vapor. (a) cloud. $T = 0°C$, $RH = 100\%$.

vapor pressure of water $= 6.11$ mb $= 611$ Pa $= 611$ N m^{-2} at $T = 0°C$.

$p = \rho R T$, where $R = 461$ J deg^{-1} kg^{-1} for H_2O, so $\rho = 4.85$ g m^{-3}

The mass absorption coeff. for pure water vapor is given as 0.5 cm^2 g^{-1}
$= 5 \times 10^{-5}$ m^2/g.

The absorption coeff for our saturated air is $\left(5\times10^{-5} \frac{m^2}{g}\right)\left(4.85 \frac{g}{m^3}\right) = 2.43 \times 10^{-4}$ m^{-1}

Absorption optical depth for water vapor:

$$\tau^*_{abs}(\text{w.v.}) = (2.43\times10^{-4} \text{ m}^{-1})(1000 \text{ m}) = 0.243$$

Absorption optical depth for droplets:

$$\tau^*_{abs}(\text{drop}) = (1-\hat{\omega}_{drop})(\tau^*_{drop}) = 10^{-4}(15) = 1.5\times10^{-3}.$$

Total optical depth $= 15.0 + 0.243 = \boxed{15.243 = \tau^*}$

single-scattering co-albedo $= \dfrac{\tau^*_{abT}}{\tau^*} = \dfrac{0.243 + 1.5\times10^{-3}}{15.243} = 0.016 = 1-\hat{\omega}.$

so $\boxed{\hat{\omega} = 0.9840}$. $\quad g = 0.85$; $1-\hat{\omega}g^2 = 0.2891.$

$$\tau^*_{\delta E} = (15.243)(.2891) = \boxed{4.41 = \tau^*_{\delta E}}$$

(b) <u>snow</u>

The water vapor occupies only the air space. The fraction of total volume <u>not</u> occupied by ice is $\left(1-\dfrac{0.4}{0.92}\right) = 0.57$.

So the absorption coeff. for our saturated air is 57% of that in the cloud: $(2.43\times10^{-4}\ \text{m}^{-1})(0.57) = 1.37\times10^{-4}\ \text{m}^{-1}$

For the 1-cm snowpack, $\tau^*_{abs}(\text{w.v.}) = (1.37\times10^{-4}\ \text{m}^{-1})(0.01\ \text{m}) = \underline{1.37\times10^{-6}}$.

$\tau^*_{abs}(\text{ice}) = (1-\hat{\omega}_{ice})\cdot\tau^* = (65.2)(10^{-3}) = \underline{6.52\times10^{-2}}$

So $\tau^*_{abs}(\text{ice}) \gg \tau^*_{abs}(\text{w.v.})$, so $\hat{\omega}$ is essentially unchanged* from problem 3. Also τ^*, $\tau^*_{\delta E}$ unchanged from problem 3. This means we can neglect absorption by water vapor when calculating radiative transfer in snow, but cannot neglect it in a cloud.

*$(1-\hat{\omega})$ changes from 1.0×10^{-3} to 1.00002×10^{-3}