

~~ready asst JS chaps 6.1-6.7~~

10 Apr 2017, Sub lecture.

MONTÉ-CARLO Method

handout:

do not read beyond p51 now

After completing the project, read pp 52-53-53a.

See if your answer was wrong; correct it if nec.

Monte Carlo is a city on the Mediterranean. What's it famous for?

- ① Grand Prix auto race
- ② Jacques Cousteau's oceanographic museum
- ③ Gambling casinos.

MC is a game of chance.

~~It is ess~~ MC method is essential for unusual geometries like inhoming clouds (Ca).

We will apply it to a simple case that can be solved in other ways, so that we can compare results.

refs (vigraf): House Arey } copies are on shelf in 400
Cashwell Everett

Should read House Arey.

Approach is alternative to RTE

RTE Eulerian ^{has the} ~~same~~ of intens field ^{varies} w loc & angle
MC Lagrangian: follow motion of indiv photons.

Method dev'd in 1970s at Los Alamos for tp of neutrons in neutron reactors.

Entered the open lit ~1959 w. Cashwell-Everett's book.

Car chooses a sequence of random numbers

r_1
 r_2
 r_3
 r_4
 $\vdots$

to determine

r_1 how far an entering photon goes before it collides (or escapes)

r_2 whether absd or scatted on collision

r_3, r_4 dirn of scatg $\Rightarrow (\theta, \phi)$

r_5 how far it goes in τ after scatg

$\vdots$



First

Specify bdryst & bdy condns. Then throw photons into medium in a certain way, and/or let them be emitted within the medium, follow their progress until they escape or are absd. Can be absorbed at walls.

~~What these & Avery call the~~
~~"Fundamental Principle" of MC~~

Suppose an event is going to occur at some value of x , bounded by a, b .

Plot ~~prob~~ ^{freq} of occurrence as fn of x .



define Cum probability $P_c(x)$ prob of ~~obtaining~~ occurrence at any value below x

~~$P_c(x)$~~

write
 $\left[P_c \text{ to disting from phase fn } P \right]$

(3)

What those & Avery call the

"Fundamental Principle" of MC :

If $p(x)dx$ is the proby of an event occuring bet x and $x+dx$
with $a \leq x \leq b$ and $\int_a^b p(x)dx = 1$

[$p(x)$ is proby ^{density}, $p(x)dx$ is proby]
(per unit increment of x)

Then

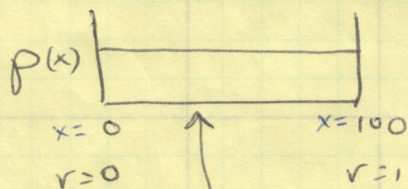
$$r = P(x) = \int_a^x p(s) ds$$

determines x as a fn of the random number r ,

$$0 \leq r \leq 1$$

so pick random numbers bet 0 & 1 will give (by this formula)
the correct ^{freq} distn of values of x .

eg. 1 Suppose $p = \text{constant}$, $a=0$, $b=100$



then if pick a random number $r=0.4$ it should point to $x=40$

In this case $p(x) = \frac{1}{100} \Rightarrow \int_0^{100} \frac{1}{100} dx = 1$

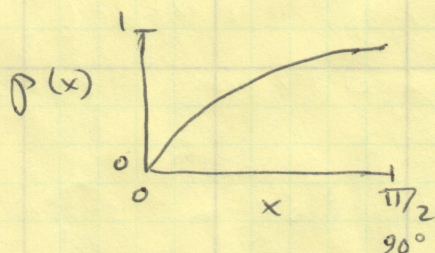
$$r = \int_0^x \frac{1}{100} ds = \frac{x}{100} ; \quad x = 100r$$

(f)

Next (eg. 2)Suppose $p(x) = \sin x$

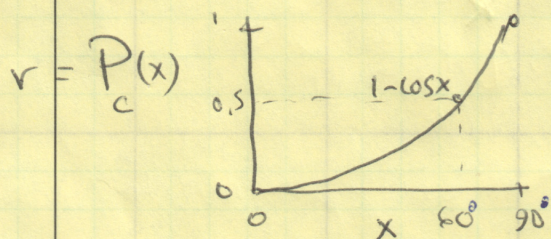
$$0 \leq x \leq \pi/2$$

Is it normalized?



$$\int_0^{\pi/2} \sin x dx = -\cos x \Big|_0^{\pi/2} = 1 \quad \text{yes.}$$

$$r = \int_0^x \sin \theta d\theta = 1 - \cos x \quad \text{plotted left}$$



$$x = \cos^{-1}(1-r)$$

$$\text{take } r = 0.5 \Rightarrow x = 60^\circ$$

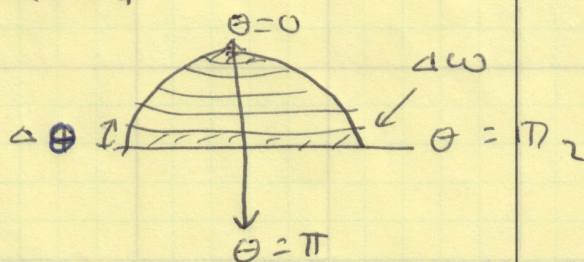
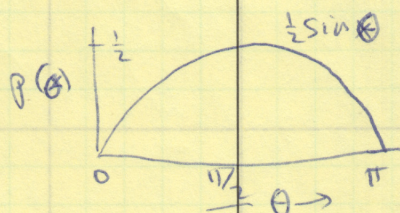
Half of random trials will give $x < 60^\circ$, half $x > 60^\circ$ Does that make sense? Not 45° because more probable at large x if $p(x) = \sin x$ (eg 3) isotropic scatty, get both θ, ϕ . First θ .~~Scatty is equally likely into any solid angle~~

what is proby density



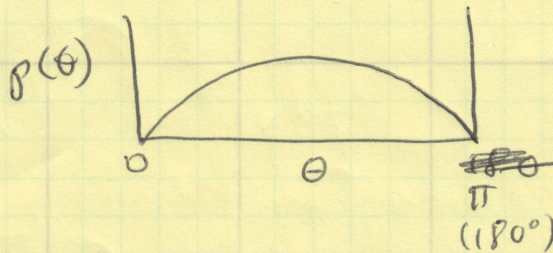
$$p(\theta) \propto \sin \theta \quad \text{because } d\omega \propto \sin \theta$$

$$d\omega = \sin \theta d\theta d\phi$$

Large solid angle $d\omega$ ~~at $\theta = \pi/2$~~
for same $d\theta$ as $\theta \rightarrow \pi/2$ ie like (eg 2) but proby const is $\frac{1}{2}$ not 1because $0 \leq \theta \leq \pi$ not $\pi/2$.

(5)

$$p(\theta) = \frac{1}{2} \sin \theta$$

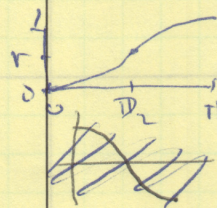


$$r = \int_0^\theta \frac{1}{2} \sin \theta d\theta = -\frac{1}{2} \cos \theta \Big|_0^\theta = \frac{1}{2} (1 - \cos \theta)$$

~~cos θ~~ $\theta = \cos^{-1} [1 - 2r]$

$r=0 \quad \theta=0$

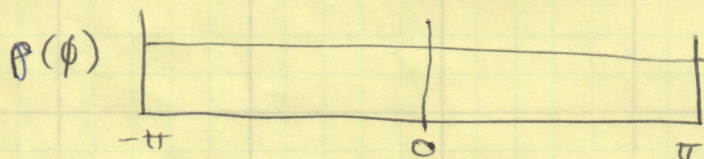
$r=1 \quad \theta=180^\circ$



will see in CBE $\theta = \cos^{-1} [2r - 1]$

This is equiv, just means r is running in reverse $1 \rightarrow 0$ stat $0 \rightarrow 1$

r	$2r-1$	$1-2r$
0	-1	1
1	1	-1



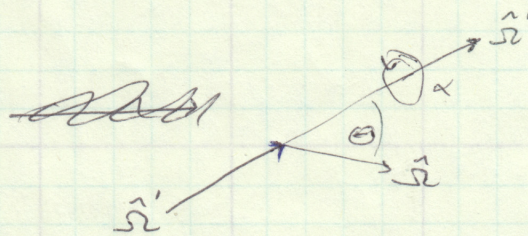
const for isotropic scattg

4. For a general

for arbitrary phase function, procedure would be to pick 2 random numbers.

$r_1 \Rightarrow \Theta$ using phase fn

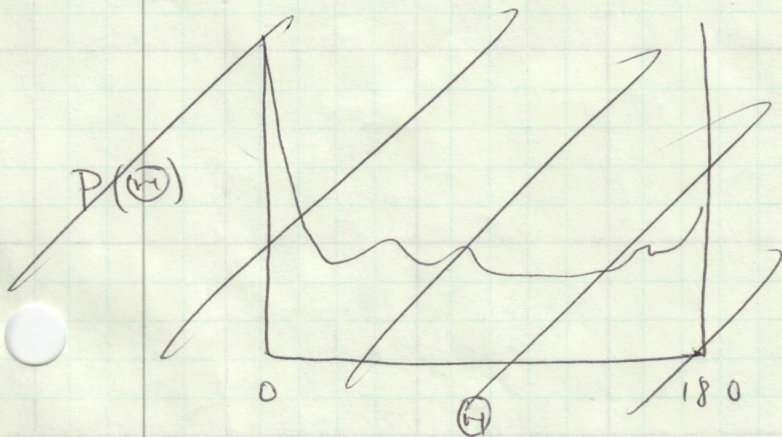
$r_2 \Rightarrow$ azimuth α about $\hat{S}'$



$$\{\alpha, \Theta, \hat{S}'\} \Rightarrow \hat{S} [= (\Theta, \Phi)]$$

$$\{\hat{S}', \Theta, \alpha\} \Rightarrow$$

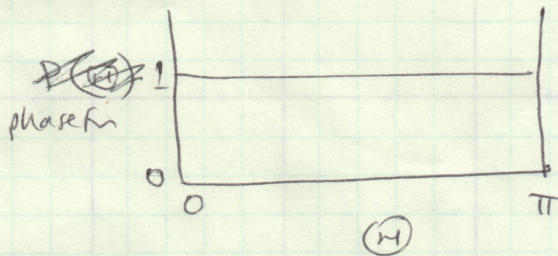
eg. for ~~the~~ water droplet in a cloud, phase fn is ~~is used~~
~~unfortunately same symbol for phase fn & prob. density~~
~~the P is phase fn.~~ Phase fn is diff from
~~this is not yet the prob. density~~



How are you normalizing the phase fn?

Are you using the ^{normalization} convention that phase fn = 1 for isotropic scatg?

first eg isotropic

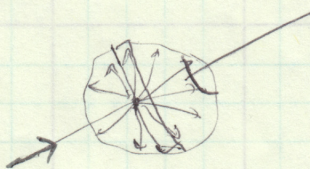


if normalized by

$$\frac{1}{4\pi} \int_{4\pi} P(\Theta) d\omega = 1$$

Then what is prob. density of Θ . $P(\Theta)$? $\neq$ phase fn

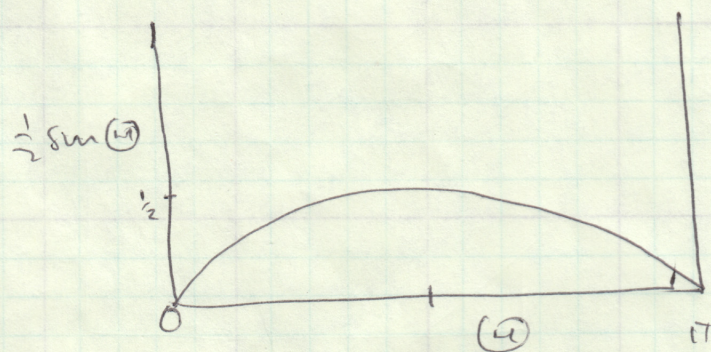
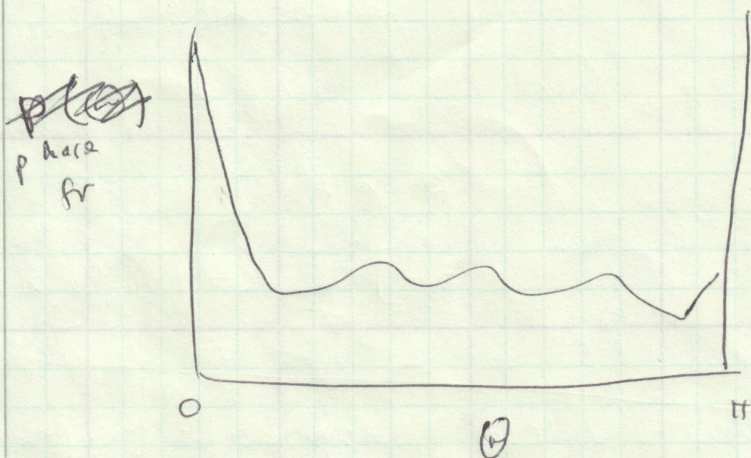
For isotropic, $P(\Theta) = \frac{1}{2} \sin \Theta$



normalized

$$\int_0^\pi P(\Theta) d\Theta = 1$$

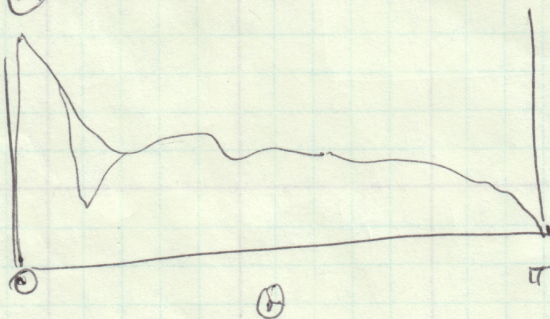
for a general phase fn eg a water droplet in a cloud.



prob density
of θ

$$p(\theta) = \frac{1}{2} P(\theta) \sin(\theta)$$

↑ phase fn



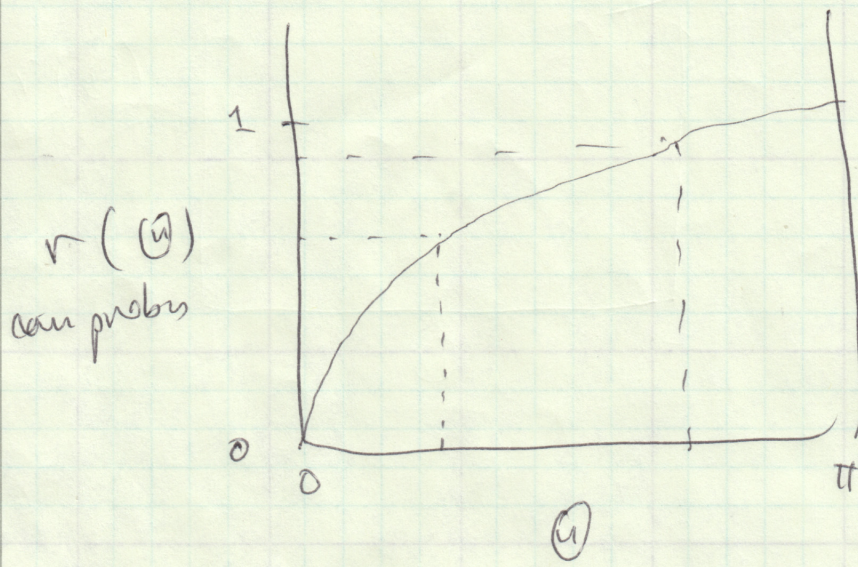
so random number $0 \leq r \leq 1$: conversion of r to θ

$$r(\theta) = \int_0^\theta \frac{1}{2} P(\varphi) \sin \varphi d\varphi$$

↑ phase fn.

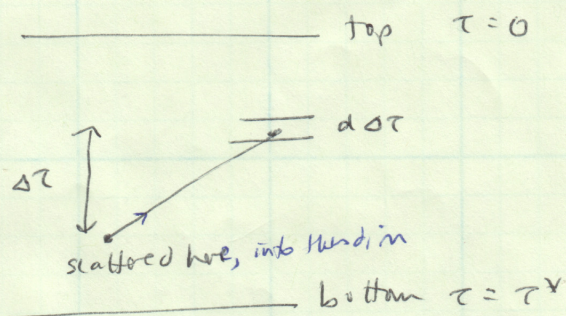
now calc r for many θ 's

cum proby



Make a look-up table. of partial sums

5. After choosing angle, next random number ^{determines} ~~chooses~~ how far ($\Delta\tau$) the photon will travel. Let's do it for plane-parallel geom.



probability density $p(\Delta\tau)$ is [prob per unit optical depth] of getting stopped at dist $\Delta\tau$ to get a prob, mult by an o.d. increment: $p(\Delta\tau) d\Delta\tau$ units are OD^{-1}

$$p(\Delta\tau) d\Delta\tau = \underbrace{\left(\text{prob of travel at least distance } \Delta\tau \right)}_{e^{-\Delta\tau/\mu}} \times \underbrace{\left(\text{prob of extinction in layer } d\Delta\tau, \text{ given that it reached this layer} \right)}$$

prob of travel at least $\Delta\tau$ $t = e^{-\Delta\tau/\mu}$

$t = 1 - \frac{d\Delta\tau}{\mu}$

extn = $1 - t = \frac{d\Delta\tau}{\mu}$

so $p(\Delta\tau) = \frac{1}{\mu} e^{-\Delta\tau/\mu}$. Is it normalized?

$$\int_0^{\infty} p(\Delta\tau, \mu) d\Delta\tau = 1. \quad \text{yes.}$$

note ∞ not τ^* . why?

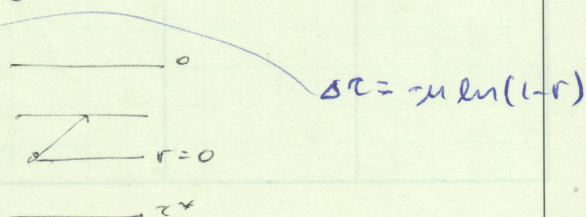
so $r = \int_0^{\Delta\tau} \frac{1}{\mu} e^{-\tau'/\mu} d\tau' = \left[1 - e^{-\Delta\tau/\mu} \right]$

what is this? is it an expl se? No! (integral over angle not opt depth)

Then solve for $\Delta\tau(r)$.

Mapping of r to $\Delta\tau$;

$r=1$ will take you outside the medium: $\Delta\tau = \infty$.



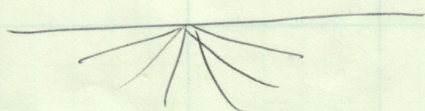
Note whether the photon escapes the medium. If so, this that comes to an end. Record which angular bin it emerges into, and record its angle. ~~Then shoot in another photon.~~



Computer follows progress of a photon until it emerges. Then shoot in another photon.

(vugraf) → read the problem.

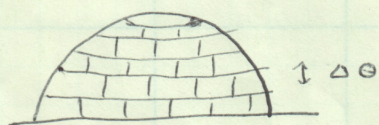
For this problem need only Θ -bins.



Don't req. dists w- ϕ . Careful on converting counts to I. Don't read pp 52, 53

until after running program.

If want to get ^{variation} ~~dist~~ of intensity ~~var~~ with both Θ, ϕ :



"Igloo".

In handout is algorithm I worked out for how to build igloo with any specified number of bins, giving them all equal solid angle (exactly equal).

(Jingood at building mather igloos. But real igloo out of snow; mine ended up looking more like a pyramid ~~than~~ than a hemisphere.

Statistical accuracy of M-C

(11)

we need thousands of photons to get ~~statistical~~ ^{accurate results,} accuracy.
If n = number of photons in a bin
Expected error $\sigma(n) = n^{1/2}$. we'll explain why

$$\text{So relative error} = \frac{\sigma(n)}{n} = \frac{1}{\sqrt{n}}$$

~~proved in handout pp 45-47.~~

$$100 \text{ photons} \pm \sqrt{100} \Rightarrow 10\% \text{ error}$$

$$10000 \pm \sqrt{10000} \Rightarrow 1\% \text{ error.}$$

$\downarrow$
100

If 2×10^6 photons total and 100 bins in ^{upper} igloo, 100 in ^{bins} lower igloo
then avg 10,000 photons / bin

avg Error ~ 1%.

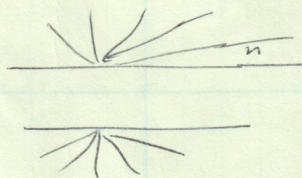
But ~~relative~~ error is different in each

~~in each~~ bin because some have more than 10,000, some less

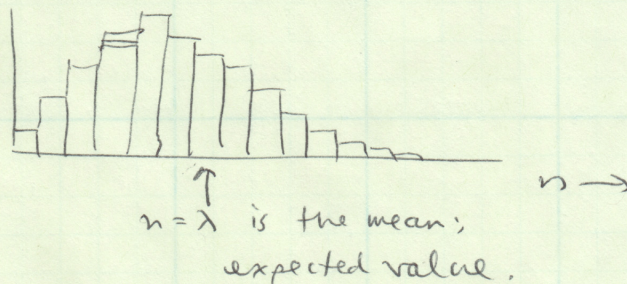
The relevant statistical model is the

Poisson distribution for counting photons. (handout pp 45-47)

if you ran your program 100 times



freq (n)
~~prob~~
 $f(n)$



$\lambda \equiv$ expected number of events (photons energy in bin i)
in a specified time period.

Poisson dist is $f(n) = \frac{\lambda^n}{n!} e^{-\lambda}$

~~proven~~ That this is the approp distn for photon count
is ~~proven~~ in Mendenhall's book (pages in handout).
Sham

properties of poisson distr.

(a) $\sum_{n=0}^{\infty} p(n) = 1$. I show in handout.

(b) Mean value $\bar{n}$ is λ $\bar{n} = \frac{\sum n \cdot p(n)}{\sum p(n)} = \lambda$

(c) Variance of n is λ ie variance = mean

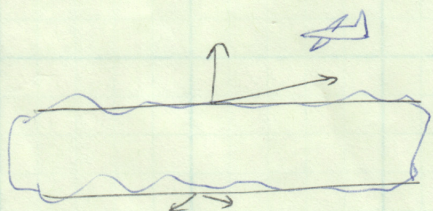
$\therefore$ standard deviation = $\sqrt{\lambda}$.

So calculate n_i from Ch' exact solution for n in bin 1.

Then in count should expect $n_i \pm \sqrt{n_i}$.

Technique to reduce computer time: BIASING (discussed in H.A.)
(not req'd for this project)

Suppose in real life you have a detector you move around to look at different angles.



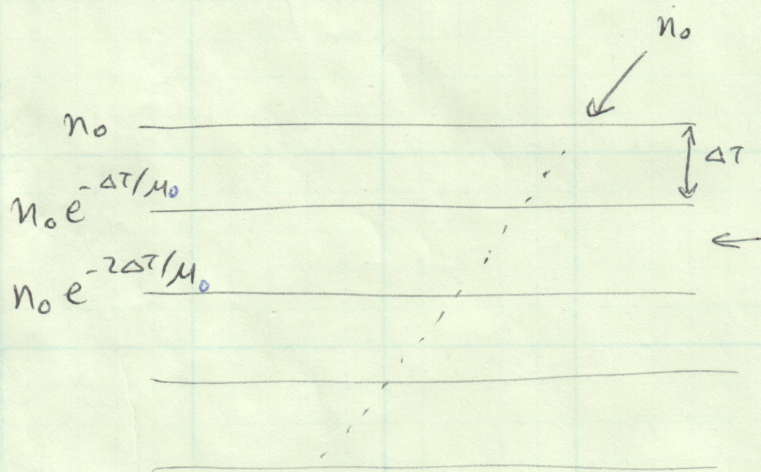
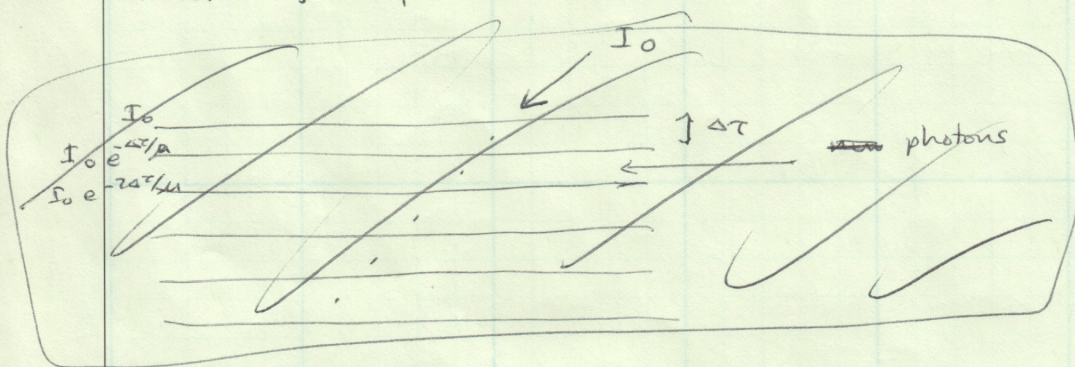
You don't need to spend so much time at angles with lots of photons: spend longer counting photons where ~~angle~~ intensity is weak. Eg if you want 1% error;

then just count until you've got 10,000 photons, record how long it took, then move to another angle. Fly fast above cloud, slow below cloud

The way this is done in the car is biasing. After the probys get more scatty into improbable angles or improbable optical depths, but then correct the results afterwards.

Eg for our cloud we'd like to spend more time counting transmitted light than reflected light, because for $\tau = 2$ refed light is brighter.

A form of biasing you could do in this project is to have photons 'born' along the path of incident beam as if they were emitted there. Start equal numbers in each layer, but weight them by the prob. ~~they would~~ that inc. photons would actually reach that layer. So each photon counts as only a ~~part~~ fraction of a photon.



number of photons undergoing first scattering in ~~the~~ layer 2 is

$$n_2 = n_0 [e^{-\Delta \tau / \mu_0} - e^{-2\Delta \tau / \mu_0}]$$

This can be computed in advance before getting random numbers.

Can ^{pre}compute for each layer how many of initial photons survive to the top of the layer but fail to survive passage of layer.

Biasing: start equal numbers in each layer but then weight them by $\frac{n_i}{n_0}$.

Another biasing - method:
weight for absorption.

Don't destroy a photon when it's absorbed; only weight it down and let the remaining partial photon continue to scatter.
Eg. if $\bar{w} = 0.8$ then after 1 collision have 0.8 of a photon remaining. follow it until it escapes. Good for optically-thick media. (otherwise need lots of photons to get any emerging from bottom).

In project you're also ~~asked~~ asked to compute the source fn $S'(\tau)$
ie. keep track of how many photons scattered in each ~~tau~~ τ -layer.
and keep track of how many times this photon has already scattered.

τ	$S_0(\tau)$	$S_1(\tau)$	$S_2(\tau)$	...	S_{total}
0-0.1	11	1			11 11
0.1-0.2	1	1			11 11
0.2-0.3					
	first single scattering	second double scattering			

Then see how accuracy compares with exact tabulated solution for emergent I [X, Y. function of χ' .]

read from assignment on vugrat.

So you will get $I(\theta)$ in 3 ways

① exact from X, Y function

② counting

③ $\int$ surface. $I = \int S e^{-\tau} \frac{d\tau}{u}$

this will be more accurate than counting because each photon counts not just once but every time it scatters

Optional extensions of project.

1. vary τ^* , vary θ_0 easy

2. $\tilde{\omega} < 1$ easy

3. keep track of ϕ also. Then more computer time

because  is too and need more photons to get signif counts/bin. 100 bins each side. only 10 bins needed each side for θ .

4. $P(\theta)$ not isotropic

MC problem what do you expect for results.

This is a direct solar beam illuminating a thin cloud at nsl λ_s .

$\tau^* \sim 20$ for Sc of eastern pacific

but scaled τ^* is ≈ 3

(of two-scattering droplets)
i.e. real cloud of $\tau^* = 20$ has same alb and mult
as cloud of isotropic scatterers with $\tau^* = 3$.

We'll see later how to do this scaling.

